Schwatt - Geometrical Treatment of Curves - 1895.

A

GEOMETRICAL TREATMENT OF CURVES

WHICH ARE

ISOGONAL CONJUGATE TO A STRAIGHT LINE WITH RESPECT TO A TRIANGLE

IN TWO PARTS

PART FIRST

BY

I. J. SCHWATT, PH.D.
UNIVERSITY OF PENNSYLVANIA

LEACH, SHEWELL, AND SANBORN
BOSTON NEW YORK CHICAGO

Norwood Press:
J. S. Cushing & Co. — Berwick & Smith.
Norwood, Mass., U.S.A.

TO

My Esteemed Colleague

PROFESSOR EDWIN S. CRAWLEY

CONTENTS OF PART FIRST.

A

GEOMETRICAL TREATMENT OF CURVES.

1. Two straight lines are said to be isogonal conjugate with respect to an angle when they form equal angles with the bisector of the angle. Let the point of intersection of three straight lines, each drawn from an angular point of a triangle, be O, then their isogonal conjugates will also intersect in one point, say O', which is called the isogonal conjugate point to the point O, with respect to the three angles of the triangle. The isogonal conjugate points to all points on a straight line with respect to a triangle will form a curve which is said to be isogonal conjugate to the straight line with respect to the triangle. If all the points on a straight line LL', Fig. 1, are joined with the angular points B, and C, of the triangle ABC, then we get two projective flat pencils which are also perspective to one another, since the intersection of the corresponding rays are collinear. The isogonal conjugate points to all points on LL', if joined to B and C, will form two projective flat pencils. The intersections of two corresponding rays are in general not on the same straight line, and are therefore not perspective; and their locus is a conic section, as is known from the fundamental laws of projective geometry.

2. The conic isogonal conjugate to a straight line with respect to a triangle will pass through the angular points of the triangle. It will pass through B and C, Fig. 1, since they are in our case the vertices of the flat pencils. For, let the point of intersection of the given line LL' with the side BC be

P_a, then BA is isogonal conjugate to BP_a; and CA is isogonal conjugate to CP_a. The point of intersection of BA and CA is the isogonal conjugate point to the point of intersection of BP_a and CP_a; or, A is the point isogonal conjugate to P_a, and the conic will, therefore, pass through A. Taking A and C, or A and B, as the vertices of the flat pencils, we have B the isogonal conjugate point to P_b, and C the isogonal conjugate point to P_c; and the conic passes also through B and C.

3. The character of the conic isogonal conjugate to the line LL' with respect to the triangle ABC depends upon the position of the line with respect to the circumcircle of the triangle ABC. We know that the point isogonal conjugate to a point on the circumcircle of the triangle is a point at infinity. In order to find the isogonal conjugate point to P on the circumcircle of ABC, we have to find the intersection of the lines isogonal conjugate to PA, PB, PC (Fig. 2), which are AA', BB', and CC'.

If AA' is isogonal conjugate to AP, then angle BAA' equals angle PAC $(=x)$. Let the isogonal conjugate to BP be BB'; then angle PBA equals angle CBB'. Since angle $CBP=x$, therefore angle PBA equals angle $CBB'=\beta-x$, denoting the whole angle B by β. It follows, therefore, that angle $B'BA$ equals x, and BB' is parallel to AA'. In the same way it may be shown that CC' is parallel to BB'. Thus, AA', BB', and CC' are all parallel to each other, and the isogonal conjugate point to P, at their intersection, is therefore at infinity.

4. If the line LL', Fig. 1, does not cut the circumcircle of the triangle, then the conic isogonal conjugate to this line with respect to the triangle has no point at infinity, and the conic will be an ellipse. If the line intersects the circle about the triangle, the conic isogonal conjugate to the line will have two points at infinity, and will, therefore, be an hyperbola. If the line touches the circumcircle of the triangle, then the conic will have one point at infinity, and will, therefore, be a parabola.

1. THE HYPERBOLA.

5. The conic isogonal conjugate to a line which cuts the circumcircle of a triangle is, with respect to this triangle, an hyperbola, which is called Kiepert's Hyperbola.

If the straight line cuts the circumcircle in P and Q, Fig. 3, then will the line AP', isogonal conjugate to AP, determine the direction of the point at infinity, *i.e.* the direction of one of the asymptotes of the hyperbola. The line AQ', isogonal conjugate to AQ, will determine the direction of the second asymptote. (The asymptotes will be parallel to these directions.) AP and AQ form the same angle as their isogonal conjugates. Since angle BAP equals angle $P'AC$, and angle QAC equals angle BAQ', therefore $\measuredangle BAP + \measuredangle CAB + \measuredangle QAC$ equals $\measuredangle P'AC + \measuredangle CAB + \measuredangle BAQ'$, or angle QAP equals angle $P'AQ'$. Since the asymptotes of the hyperbola isogonal conjugate to PQ with respect to the triangle ABC are parallel to AP' and AQ', therefore the angle between the asymptotes of the hyperbola will be equal to the angle between AP and AQ. If the line PQ passes through the centre of the circle, then will the angle formed by AP and AQ be a right angle; the asymptotes of the hyperbola will be parallel to these lines, hence perpendicular to each other; and the hyperbola will be equilateral.

6. We shall now examine the hyperbola isogonal conjugate to Brocard's Diameter (produced). Brocard's diameter is the line joining the centre of the circumcircle and Grebe's point of the triangle. Grebe's point has the property that its distances from the sides of the triangle are to each other as the respective sides; and the sum of the squares of these is, for the triangle, a minimum. As Brocard's diameter passes through the centre of the circumcircle, the conic isogonal conjugate to it is an equilateral hyperbola. We shall locate points with respect to the triangle isogonal conjugate to points on Brocard's diameter. All these points will be on the equilateral hyperbola.

Let the points of intersection of Brocard's diameter and the three sides of the triangle be P_a, P_b, and P_c; then, as was

shown (**2**), A, B, and C are points isogonal conjugate to P_a, P_b, and P_c, respectively. A, B, and C are, therefore, on the equilateral hyperbola. The median point E is isogonal conjugate to Grebe's point K, which is one extremity of Brocard's diameter; E is therefore also a point on the equilateral hyperbola isogonal conjugate to Brocard's diameter with respect to the triangle.

From the similar triangles ABH_a and AUC, Fig. 4, it follows that AH_a is isogonal conjugate to AU. Similarly it can be shown that the altitude from B is isogonal conjugate to the diameter through B; and the altitude from C is isogonal conjugate to the diameter through C. The orthocentre, H, therefore, is isogonal conjugate to M, the centre of the circumcircle of the triangle. And since M is on Brocard's diameter, therefore is H on the equilateral hyperbola.

7. If A_1, B_1, and C_1 are the angular points of Brocard's triangle, then the lines AA_1, BB_1, and CC_1 are concurrent. Let the point of concurrency be D (Fig. 6). This may be proved as follows: —

Let (Fig. 5, Plate 2) M_a, M_b, M_c be the middle points of the sides BC, AC, AB of the triangle ABC. Let the tangent drawn at C to the circumcircle of the triangle meet the symmedian line (isogonal conjugate to the median line) through A at the point K', — we know that the tangent to the circumcircle at B will meet the symmedian line through A also in K' — then $K'M_a$ is perpendicular to BC. But since A_1M_a is also perpendicular to BC (A_1 is an angular point of Brocard's triangle), therefore A_1, M_a, and K' are collinear. It is known that $C\{K'K, AB\}$ is an harmonic pencil, and $\{K'K, AB\}$ is an harmonic range. If M_a be taken as a vertex of the harmonic pencil, then $M_a\{KK', AB\}$ is also an harmonic pencil; and as A_1K is parallel to M_aB, therefore its harmonic conjugate will bisect A_1K, or AM_a bisects A_1K. In the same way it may be proved that BM_b bisects B_1K, and CM_c bisects C_1K. Let the point of intersection of AA_1 and BC be A_{1a}; then since AM_a bisects A_1K in the triangle AK_aA_{1a}, therefore AM_a will also

bisect $K_a A_{1a}$, or $A_{1a}C$ equals $K_a B$. Therefore AA_{1a} and AK_a are isotomic conjugate lines. If the intersection of BB_1 and AC be B_{1b}, and of CC_1 and AB be C_{1c}, then BB_{1b} and BK_b are isotomic conjugates, as are also CC_{1c} and CK_c. Now since AK_a, BK_b, and CK_c are concurrent (at K), therefore their isotomic conjugates AA_{1a}, BB_{1b}, CC_{1c} are also concurrent, the point of concurrency being D.

8. Let (Fig. 6, Plate 1) M_a, M_b, M_c be the middle points of the three sides BC, AC, AB of the triangle ABC; and M'_a, M'_b, M'_c be the middle points of the sides B_1C_1, A_1C_1, A_1B_1 of Brocard's triangle; then will S, the point of intersection of $M_aM'_a$, $M_bM'_b$, and $M_cM'_c$, be isogonal conjugate to S', the point of intersection of AM'_a, BM'_b, and CM'_c. In order to prove this it will be necessary to give a few properties of the triangle.

9. If A_1M_a (Fig. 6), which is perpendicular to BC (**7**), be produced below BC, so that A_2M_a is equal to A_1M_a, then $AC_1A_2B_1$ is a parallelogram.

We have $\quad \measuredangle C_1BA_2 = \measuredangle C_1BC + \measuredangle CBA_2;$

but since $\quad \measuredangle CBA_2 = \measuredangle A_1BC = \delta$ (Brocard's angle),

therefore $\quad \measuredangle C_1BA_2 = \measuredangle C_1BC + \measuredangle A_1BC.$

But $\quad \measuredangle A_1BC = \measuredangle C_1BA = \delta,$

hence $\quad \measuredangle C_1BA_2 = \measuredangle C_1BC + \measuredangle C_1BA = \measuredangle \beta.$

The triangle A_1BM_a is similar to triangle C_1BM_c (since they are right triangles having $\measuredangle A_1BC$ equal to $C_1BA =$ Brocard's angle).

Therefore
$$A_1B : C_1B = BM_a : BM_c = \frac{a}{2} : \frac{c}{2} = a : c.$$

But $\quad A_1B = A_2B,$

whence $\quad A_2B : C_1B = a : c;$

and since the included angle between A_2B and A_2C equals the

angle included between a and c, therefore triangle BC_1A_2 is similar to the triangle ABC. In the same way it may be proved that triangle B_1CA_2 is also similar to the triangle ABC, or triangle C_1BA_2 is similar to triangle B_1CA_2. But since A_2B equals A_2C, therefore the last-mentioned triangles are not only similar, but equal, and consequently B_1A_2 equals C_1B. But also C_1B equals C_1A, hence B_1A_2 equals C_1A, and C_1A_2 equals B_1C, equals B_1A; or the figure $AC_1A_2B_1$ is a parallelogram.

10. The triangle ABC and its Brocard triangle have the same median point, E. Since $AC_1A_2B_1$ (Fig. 6) is a parallelogram, the diagonals AA_2 and B_1C_1 bisect each other at the point M'_a. $A_1M'_a$ is a median line in the triangle $A_1B_1C_1$, as well as in the triangle AA_1A_2. A second median line in the triangle AA_1A_2 is AM_a (since $A_1M_a = A_2M_a$); we have, therefore, that A_1E equals $2\,EM'_a$, and AE equals $2\,EM_a$. But AM_a is also a median line in the triangle ABC, and therefore E is the median point in $A_1B_1C_1$, as well as in ABC. By means of these last two properties, we may proceed to prove that $M_aM'_a$, $M_bM'_b$, and $M_cM'_c$ are concurrent.

Since E is the common median point of the triangles ABC and $A_1B_1C_1$, therefore,

$$AE : EM_a = 2 : 1 = A_1E : EM'_a,$$

or SM'_a is parallel to AA_1, as the triangles $M_aEM'_a$ and AEA_1 are similar, and therefore angle M'_aM_aE equals angle A_1AM_a. In the same manner we may show $M_bM'_b$ parallel to BB_1, and $M_cM'_c$ parallel to CC_1; and since triangle ABC is similar to triangle $A_1B_1C_1$, therefore triangle $M_aM_bM_c$ is similar to triangle $M'_aM'_bM'_c$. Triangle $A_1B_1C_1$ evidently has the same relative position with respect to ABC as triangle $M'_aM'_bM'_c$ has with respect to $M_aM_bM_c$. Therefore, since AA_1, BB_1, CC_1, intersect in one point, D, their parallels, $M_aM'_a$, $M_bM'_b$, $M_cM'_c$, will intersect in one point. This point, which is the centre of perspective of the triangles $M_aM_bM_c$ and $M'_aM'_bM'_c$, may be denoted by S.

11. The triangles ABC and $M_aM_bM_c$ are perspective, the centre of perspective being E. The line joining the two perspective points, D and S, with respect to the triangles ABC and $M_aM_bM_c$ must pass through the centre of perspective, E. Therefore S, E, and D are collinear, and $DE : ES = 2 : 1$.

12. The lines AM'_a, BM'_b, CM'_c are concurrent, the point of concurrency being S', which is isogonal conjugate to S (**10**) (Fig. 6). O, O', B_1, and C_1 are concyclic, therefore the lines OO' and B_1C_1 are antiparallel with respect to A. Triangles AOO' and AB_1C_1 are, therefore, similar. Since AS is a median line in the triangle AOO', and AM'_a is a median line in the triangle AB_1C_1, therefore the lines AS and AM'_a are isogonal conjugate lines. Similarly, BS and BM'_b are isogonal conjugate, as are also CS and CM'_c. But AS, BS, and CS are concurrent, hence their isogonal conjugates AM'_a, BM'_b, and CM'_c are concurrent, and the point of concurrency is S'. Therefore S' is the isogonal conjugate point to S, which is on Brocard's diameter, and S' is a point on the equilateral hyperbola.

13. S is the middle point of OO'. In order to prove this important property, we will first find the distances of O and O' from the vertices, and of O, O', and D from the sides of the triangle ABC.

14. If KK_a (Fig. 7), KK_b, and KK_c represent the distances of K from the three sides of the triangle, then

$$KK_a = \frac{2a\cdot\Delta}{a^2+b^2+c^2},\quad KK_b = \frac{2b\cdot\Delta}{a^2+b^2+c^2},\quad KK_c = \frac{2c\cdot\Delta}{a^2+b^2+c^2},$$

where Δ represents the area of the triangle. We have C_1M_c equal to KK_c, since KC_1 is parallel to AB, and C_1M_c and KK_c are each perpendicular to AB (**7**). We easily obtain

$$\begin{aligned} AC_1 &= \sqrt{\frac{c^2}{4} + \overline{KK_c}^2} \\ &= \frac{c}{a^2+b^2+c^2}\sqrt{a^2b^2+a^2c^2+b^2c^2}. \end{aligned}$$

Whence, also, $$\frac{C_1M_c}{C_1A}=\frac{2\Delta}{\sqrt{a^2b^2+a^2c^2+b^2c^2}}.$$

Let G (Fig. 7) be the foot of the perpendicular from B to AO, then in the similar right triangles AC_1M_c and ABG ($\measuredangle C_1AM_c=\measuredangle BAG=\delta$)

$$\frac{BG}{BA}=\frac{C_1M_c}{C_1A}=\frac{2\Delta}{\sqrt{a^2b^2+a^2c^2+b^2c^2}},$$

whence $$BG=\frac{2c\cdot\Delta}{\sqrt{a^2b^2+a^2c^2+b^2c^2}}.$$

Angle BOA equals $180-\beta$, and $BOG=\beta$, where β is angle ABC of the triangle ABC. The triangles BOG and BAH_a are similar, and

$$\frac{BO}{BG}=\frac{c}{h_a}=\frac{ca}{2\Delta},\ (h_a=\text{altitude } AH_a),$$

or $$BO=BG\cdot\frac{ca}{2\Delta}=\frac{ac^2}{\sqrt{a^2b^2+a^2c^2+b^2c^2}}.$$

A similar consideration gives

$$CO'=\frac{ab^2}{\sqrt{a^2b^2+a^2c^2+b^2c^2}}.$$

Again, BOC and $AO'C$ are similar triangles,

therefore $$\frac{AO'}{BO}=\frac{b}{a},$$

and $$AO'=BO\cdot\frac{b}{a}=\frac{bc^2}{\sqrt{a^2b^2+a^2c^2+b^2c^2}}.$$

15. Let OO_a, OO_b, OO_c, $O'O'_a$, $O'O'_b$, $O'O'_c$ be the distances of O and O' from the sides of the triangle ABC.

Triangle BOO_a is similar to triangle AC_1M_c, hence

$$\frac{OO_a}{BO}=\frac{C_1M_c}{C_1A}=\frac{2\Delta}{\sqrt{a^2b^2+a^2c^2+b^2c^2}},$$

therefore $$OO_a=\frac{2ac^2\cdot\Delta}{a^2b^2+a^2c^2+b^2c^2}.$$

Similarly, $$OO_b = \frac{2\,a^2b \cdot \Delta}{a^2b^2 + a^2c^2 + b^2c^2},$$

and $$OO_c = \frac{2\,b^2c \cdot \Delta}{a^2b^2 + a^2c^2 + b^2c^2}.$$

Again, triangle $CO'O'_a$ is similar to triangle AC_1M_c, whence

$$\frac{O'O'_a}{CO'} = \frac{C_1M_c}{C_1A} = \frac{2\,\Delta}{\sqrt{a^2b^2 + a^2c^2 + b^2c^2}},$$

whence $$O'O'_a = \frac{2\,ab^2 \cdot \Delta}{a^2b^2 + a^2c^2 + b^2c^2}.$$

Similarly, $$O'O'_b = \frac{2\,bc^2 \cdot \Delta}{a^2b^2 + a^2c^2 + b^2c^2},$$

$$O'O'_c = \frac{2\,a^2c \cdot \Delta}{a^2b^2 + a^2c^2 + b^2c^2}.$$

16. Let X be the middle point of OO', then will its distance XX_a from BC be

$$XX_a = \frac{OO_a + O'O'_a}{2} = \frac{2\,\Delta\,ac^2 + 2\,\Delta\,ab^2}{2[a^2b^2 + a^2c^2 + b^2c^2]}$$
$$= \frac{\Delta \cdot a(b^2 + c^2)}{a^2b^2 + a^2c^2 + b^2c^2}.$$

Let A_1A_{1b} and A_1A_{1c} be the distances of A_1 from the sides AC and AB of the triangle ABC; then $A_1A_{1b}C$ and COO_a are similar triangles. Therefoie,

$$\frac{A_1A_{1b}}{OO_a} = \frac{A_1C}{OC}. \tag{α}$$

Triangles A_1M_aC and OO_bC are similar,

whence $$\frac{OO_b}{A_1M_a} = \frac{OC}{A_1C}. \tag{β}$$

Multiplying (α) and (β), we obtain, after reduction,

$$\frac{OO_a}{OO_b} = \frac{A_1A_{1b}}{A_1M_a} = \frac{c^2}{ab},$$

or $$\frac{A_1A_{1b}}{A_1M_a} = \frac{c^3}{abc}. \tag{1}$$

In a similar manner we may have,

$$\frac{A_1A_{1c}}{A_1M_a} = \frac{b^3}{abc}, \qquad (2)$$

whence

$$\frac{A_1A_{1b}}{A_1A_{1c}} = \frac{c_3}{b_3}.$$

Let DD_a, DD_b, DD_c, be the distances of D from the respective sides of the triangle; then from the similar triangles DAD_b and A_1AA_{1b}, and DAD_c, and A_1AA_{1c}, we have

$$\frac{DD_b}{A_1A_{1b}} = \frac{AD}{AA_1} \text{ and } \frac{DD_c}{A_1A_{1c}} = \frac{AD}{AA_1},$$

and by combining these equalities we get

$$\frac{DD_b}{DD_c} = \frac{A_1A_{1b}}{A_1A_{1c}} = \frac{c^3}{b^3}. \qquad (1)$$

Similarly,

$$\frac{DD_b}{DD_a} = \frac{a^3}{b^3}, \qquad (2)$$

$$\frac{DD_c}{DD_a} = \frac{a^3}{c^3}. \qquad (3)$$

Now, $a.DD_a + b.DD_b + c.DD_c = 2\,\Delta$.

If we find from (1), (2), (3) values for DD_b and DD_c in terms of DD_a, we may put

$$DD_a = \frac{2\,b^2c^2 \cdot \Delta}{a[a^2b^2 + a^2c^2 + b^2c^2]}.$$

Likewise,

$$DD_b = \frac{2\,a^2c^2 \cdot \Delta}{b[a^2b^2 + a^2c^2 + b^2c^2]}.$$

$$DD_c = \frac{2\,a^2b^2 \cdot \Delta}{c[a^2b^2 + a^2c^2 + b^2c^2]}.$$

The distances of E from the sides of the triangles are

$$EE_a = \tfrac{1}{3}h_a = \frac{2\,\Delta}{3\,a};\ EE_b = \frac{2\,\Delta}{3\,b};\ EE_c = \frac{2\,\Delta}{3\,c}.$$

Since $DE = 2\,SE$ (Fig. 7), therefore,

$$EE_a - DD_a = 2(SS_a - EE_a),$$

or
$$SS_a = \tfrac{1}{2}(3\,EE_a - DD_a);$$

i.e.
$$SS_a = \frac{\Delta \cdot a(b^2 + c^2)}{a^2b^2 + a^2c^2 + b^2c^2}.$$

But this was the value found for the length of XX_a; hence X and S coincide, and X or S is the middle point of OO'.

The point E is also the centre of gravity or median point of the triangle DOO'.

17. The point D is isogonal conjugate to D', which is the pole of the line joining the two Brocard points, with respect to Brocard's circle. (Fig. 8.)

Let G (Fig. 8) be the fourth point of the harmonic range of which the other three are: one angular point of Brocard's triangle, the middle point of the side opposite this angle, and the median point of the triangle. Then $\{A_1M'_a, EG\}$ shall represent the harmonic range, and $A\{A_1M'_a, EG\}$ is an harmonic pencil.

We have (**10**), $\quad A_1E : EM'_a = EG : A_1E,$

or
$$2 : 1 = EG : A_1E.$$

But by Euler's proposition we have

$$2 : 1 = EH : EM.$$

(H is the orthocentre, and M is circumcentre of the triangle ABC).

Wherefore, $\quad EG : A_1E = EH : EM;$

and hence $\quad GH$ is parallel to A_1M.

But as AH is perpendicular to BC, therefore will GH also be perpendicular to BC, and the point G is on the altitude from A to BC. AG and AH coincide, and since $A\{A_1M'_a, EG\}$ is an harmonic pencil, therefore $A\{A_1M'_a, EH\}$ will be an harmonic pencil. Now, since the isogonal conjugate lines corresponding to the rays of this pencil form like angles with

each other, the pencil formed by these isogonal conjugates will have the same cross-ratio, and will therefore be harmonical. Taking, then, for AM'_a, AE, and AH, their isogonal conjugates AS, AK, AM respectively, and letting the isogonal conjugate to AD be AD', then $A\{A_1S, KD'\}$ is an harmonic pencil, cutting Brocard's diameter in the harmonic range $\{MS, KX\}$. In the same way it may be shown that $B\{MS, KX\}$ and $C\{MS, KX\}$ are harmonic pencils. It follows at once that X coincides with D', and is on Brocard's diameter.

Furthermore, MK is perpendicular to and bisects OO' at the point S, and D' is the harmonic conjugate to S with respect to M and K; hence, from the relations of poles and polars, it is evident that D' is the pole of OO' with respect to Brocard's circle.

DH is parallel to MD'.

For, $\qquad DE : ES = 2 : 1,$

and by Euler's proposition,

$$HE : EM = 2 : 1.$$

Therefore, $\qquad DE : ES = HE : EM,$

or $\qquad DE : HE = ES : EM,$

and, since the angles included by these lines are vertical, it follows that HD is parallel to SM. But SM and $D'M$ are on the same straight line; therefore HD is parallel to $D'M$.

18. Tarry's point, N (Fig. 15, Plate 2), is isogonal conjugate to the point at infinity on Brocard's diameter. N is the point of intersection of the perpendiculars let fall from the angular points of the triangle ABC to the sides B_1C_1, A_1C_1, and A_1B_1 of Brocard's triangle. It is easy to see that N is on the circumference of the circumcircle of ABC. Suppose the perpendiculars from A to B_1C_1, and from B to A_1C_1, to intersect at N. Since the sides of the angle ANB are perpendicular to the sides of the angle $A_1C_1B_1$, therefore is ANB equal to $180 - A_1C_1B_1$. But the angle $A_1C_1B_1$ equals angle ACB. Therefore is ANB equal to $180 - ACB$, or N is on the circumcircle of triangle ABC.

19. N has the same position with respect to ABC_1 as M has with respect to $A_1B_1C_1$; that is, angle NAB equals angle B_1A_1M. Since NA is perpendicular to MC_1, therefore angle NAB equals angle B_1C_1M; but angle B_1C_1M equals angle B_1A_1M, since the points are concyclic. Therefore the points are similarly situated.

20. Suppose AX drawn isogonal conjugate to AN; that is, angle $NAB =$ angle XAC. We have seen (**19**) that NAB is equal to MA_1B_1, which is equal to B_1KM, or angle NAB equals $B_1KM = XAC$. Now as B_1K is parallel to AC, therefore MK must be parallel to XA, or the line which is isogonal conjugate to AN runs parallel to Brocard's diameter.

In like manner we may show that BY, the isogonal conjugate to BN, and CZ, the isogonal conjugate to CN, are also parallel to Brocard's diameter. Since the lines isogonal conjugate to AN, BN, CN, are all parallel to Brocard's diameter, it easily follows that N is isogonal conjugate to a point at infinity in the direction of Brocard's diameter (**3**). Thus N will also be on the hyperbola isogonal conjugate to Brocard's diameter with respect to the triangle.

21. Let Z' be the point isogonal conjugate to Z, the centre of Brocard's circle (Fig. 15). It can be proved that Z' is on the line joining H with S. The points A, B, C, E, H, S', Z', N, are isogonal conjugate to the points $P_a, P_b, P_c, K, M, S, Z$, and the point at infinity on Brocard's diameter respectively. The points A, B, C, E, H, S', Z', and N, are therefore on the hyperbola isogonal conjugate to Brocard's diameter with respect to the triangle. Before proceeding to examine this hyperbola, we shall give a few properties of the equilateral hyperbola.

22. (1) Any chord of an equilateral hyperbola appears the extremities of a diameter of the hyperbola either under the same (when the diameter without being produced meets the chord), or under supplementary angles (when the diameter must be produced to meet the chord).

Let AA_1 (Fig. 14, Plate 1) be a diameter and BC a chord of an equilateral hyperbola whose asymptotes are the lines α and α_1; then it is to be proved that the angle BAC equals the angle BA_1C. Let AB meet the asymptotes α and α_1 at the points D and D_1, and let AC meet them at E and E_1, and A_1B at F and F_1. Draw AG parallel to A_1B, then triangle AMG is equal to triangle A_1MF (since AM equals A_1M and the angles are equal), therefore MF is equal to MG, and AG equals A_1F equals BF_1 (since A_1 and B are points on the hyperbola, and the intercepts of any chord between the curve and its asymptotes are equal). But AG runs parallel to BF_1, therefore ABF_1G is a parallelogram, and AB equals GF_1. But MF equals MG, and MF_1 is perpendicular to FG, hence FF_1 equals F_1G, and angle FF_1M equals angle MF_1G.

If BK be drawn perpendicular to the asymptote α, then

$$\measuredangle FF_1M = \measuredangle BF_1K = \measuredangle MF_1G = \measuredangle KD_1B,$$

and since $$\measuredangle BF_1K = \measuredangle BD_1K,$$

and BK is perpendicular to F_1D_1, therefore,

$$BF_1 = BD_1 = AD = AG.$$

In the same way, if AH be drawn parallel to A_1C, it can be shown that AE equals AH. But AD was equal to AG.

Therefore, $$\measuredangle DAE = \measuredangle HAG;$$

but $$\measuredangle DAE = \measuredangle BAC,$$

and $$\measuredangle HAG = \measuredangle BA_1C;$$

whence, $$\measuredangle BAC = \measuredangle BA_1C,$$

and the proposition is proved.

23. The bisectors of the angle and its adjacent supplementary angle formed by the lines joining any point of an equilateral hyperbola with the extremities of a diameter of the hyperbola are parallel to the asymptotes.

Let B (Fig. 14) be a point of the hyperbola, and A and A_1 the extremities of a diameter; it is to be proved that the bisec-

tor of the angle ABA_1 is parallel to α_1, and the bisector of the adjacent supplementary angle A_1BD_1 is parallel to α.

The sides of the angles FF_1G and FBA are parallel, and therefore the angles are equal. The bisector of the angle FF_1G is the asymptote α_1, therefore the bisector of angle FBA will be parallel to α_1, and the bisector of its supplementary adjacent angle F_1BD_1 will be parallel to the perpendicular to this line; that is, it will be parallel to the asymptote α.

24. The angles which the sides of Brocard's triangle form with the respective homologous sides of the given triangle are equal to the angles which Brocard's diameter forms with the radii of the circumcircle drawn to the vertices of the given triangle.

We are to prove that

$$(BC,\ B_1C_1) = \measuredangle\, AMK,$$
$$(AC,\ A_1C_1) = \measuredangle\, BMK,$$
$$(AB,\ A_1B_1) = \measuredangle\, CMK.$$

Let the point of intersection of AM and Brocard's circle be denoted by A'' (Fig. 15); then it is to be proved that

$$(BC,\ B_1C_1) = \measuredangle\, A''MK.$$

Angle AMM_b equals $\measuredangle\,\beta$, and angle A_1MM_b equals $\measuredangle\,\gamma$ (since A_1M is perpendicular to AC); therefore,

$$\measuredangle\, AMM_b - \measuredangle\, A_1MM_b = \measuredangle\, AMA_1 = \measuredangle\, A''MA_1 = \measuredangle\,\beta - \measuredangle\,\gamma.$$

But $\qquad \measuredangle\, A''MA_1 = \measuredangle\, A_1B_1A''$,

and therefore, $\measuredangle\, A_1B_1A'' = \measuredangle\,\beta - \measuredangle\,\gamma$.

Again, $\qquad \measuredangle\, A''B_1C_1 = \measuredangle\, A_1B_1C_1 - \measuredangle\, A_1B_1A''$

$$= \measuredangle\,\beta - (\measuredangle\,\beta - \measuredangle\,\gamma) = \measuredangle\,\gamma = \measuredangle\, A_1C_1B_1;$$

that is, $\qquad \measuredangle\, A''B_1C_1 = \measuredangle\, A_1C_1B_1$,

and $\qquad \measuredangle\, A''B_1C_1 = \measuredangle\, A''A_1C_1$,

hence $\qquad \measuredangle\, A_1C_1B_1 = \measuredangle\, A''A_1C_1$;

whence the line $A''A_1$ is parallel to BC. The angle formed by $A''A_1$ and A_1K (or $\measuredangle\, A''A_1K$) is therefore equal to the

angle formed by BC and B_1C_1. But since A_1 and M are on Brocard's circle,

$$\measuredangle A''A_1K = \measuredangle A''MK,$$

or

$$\measuredangle A''MK = \measuredangle (BC, B_1C_1).$$

A similar course of reasoning will establish the truth as regards the other angles.

25. The line joining the orthocentre H with Tarry's point N is a diameter of the equilateral hyperbola isogonal conjugate to Brocard's diameter with respect to the triangle ABC.

It has been proved that any chord of an equilateral hyperbola appears the extremities of a diameter of the hyperbola under equal or supplementary angles (**22**). Suppose NH is a diameter of the hyperbola isogonal conjugate to Brocard's diameter with respect to the triangle ABC (Fig. 15); then must the chords AB, AC, BC, DA, etc., appear as well N as H under the same or under supplementary angles. We shall prove in the two following statements:

$$(1)\quad \measuredangle AHC = 180 - \measuredangle ANC,$$

$$(2)\quad \measuredangle AHD = 180 - \measuredangle AND.$$

From Fig. 15 we see

$$\measuredangle AHC = 180 - \{\measuredangle HAC + \measuredangle HCA\};$$

but $\measuredangle HAC = 90 - \measuredangle \gamma$ (*i.e.* $\measuredangle C$ of triangle ABC),

and $\measuredangle HCA = 90 - \measuredangle \alpha$;

hence
$$\measuredangle AHC = 180 - \{90 - \measuredangle \gamma + 90 - \measuredangle \alpha\}$$
$$= \measuredangle \gamma + \measuredangle \alpha = 180 - \measuredangle \beta.$$

Also $\measuredangle ANC = \measuredangle ABC = \measuredangle \beta$,

therefore $\measuredangle AHC = 180 - \measuredangle ANC.$

Now, H, D', and N are on the same straight line. For, draw through D' a line parallel to HA, and let this parallel intersect AN at the point H'', then A, H'', D', and M are concyclic. This may be proved as follows: $D'H''$ is parallel to AH, and since AH is perpendicular to BC, therefore $D'H''$ is also perpendicular to BC. Since NH'' is perpendicular to B_1C_1 by

construction, it follows, therefore, that $NH''D'$ is an angle equal to the angle formed by BC and B_1C_1, which was equal to the angle AMK.

$$\measuredangle AH''D' = 180 - \measuredangle NH''D' = 180 - \measuredangle AMK;$$

and since K is on MD', therefore angle $AH''D'$ equals $180 - \measuredangle AMD'$; or, the points A, H'', D', and M are concyclic, and $\measuredangle D'H''M = \measuredangle MAD'$.

Since M and H, and D and D' are isogonal conjugate points, therefore will the angle MAD' equal angle HAD. But $\measuredangle MAD'$ was equal to $\measuredangle MH''D'$, hence $\measuredangle HAD$ equals $\measuredangle MH''D'$. Since $D'H''$ is parallel to HA, MD' is parallel to DH, $\measuredangle AHD$ equals $\measuredangle H''D'M$, and the points N, M, D, and N, H'', A, are collinear. Therefore the triangles ADH and $H''MD'$ are similar and similarly situated, the centre of similitude being N. The line joining the two corresponding vertices D' and H of the triangles ADH and $H''MD'$ must pass through the centre of similitude, N. Therefore the points H, D', and N are collinear.

26. To prove that $\measuredangle AHD = 180 - \measuredangle AND$. Angle $AH''D'$ is equal to $180 - \measuredangle AMK$,

therefore $$\measuredangle H''AM + \measuredangle H''D'M = 180.$$

In the triangle ANM, NM equals AM, being equal radii,

hence $$\measuredangle H''AM = \measuredangle NAM = \measuredangle ANM,$$

and thus $$\measuredangle ANM + \measuredangle AHD = 180.$$

Since N, M, and D are collinear (**25**) therefore,

$$\measuredangle ANM = \measuredangle AND,$$

and $$\measuredangle AND + \measuredangle AHD = 180,$$

or $$\measuredangle AHD = 180 - \measuredangle AND.$$

D is therefore a point on the hyperbola, the diameter of which is NH. The centre of the hyperbola is the middle point W of NH. The centre of the hyperbola is a point on Feuerbach's circle of the triangle ABC.

We know that the centre, F, of Feuerbach's circle is the middle point of the line joining the centre of the circumcircle

of the triangle, and the orthocentre. F is the middle point of HM, and W is the middle point of NH, therefore FW is equal to one-half of MN, or one-half the radius of the circumcircle. W is, therefore, on Feuerbach's circle.

27. Let the asymptotes of the hyperbola be α and α_1, they are then chords Wm, and Wm_1 of Feuerbach's circle, m and m_1 being the extremities of a diameter of Feuerbach's circle parallel to DH. Since D and H are points on the hyperbola, therefore will the line joining the middle point U of HD and the centre of the hyperbola bisect all chords parallel to HD, and will pass through the centre of Feuerbach's circle. But since the intercepts of a chord between the hyperbola and its asymptotes are equal, therefore UU_1 equals UU_2. If through the centre of Feuerbach's circle is drawn a parallel to HD until it intersects the asymptotes in m_1 and m_2, then Fm_1 equals Fm_2. It is easy to see that m_1 and m_2 are the extremities of a diameter of Feuerbach's circle. Since W is the centre of the hyperbola and at the same time on Feuerbach's circle, m_1m_2 is a secant line through the centre of Feuerbach's circle, and the angle m_1Wm_2 equals a right angle, therefore m_1 and m_2 must be the extremities of a diameter, and m_1 and m_2 are on Feuerbach's circle.

In order to understand the relations of the asymptotes, it will be necessary to examine some properties of the triangle, especially those of Simson's line.

SIMSON'S LINE.

28. If from a point P (Fig. 9), on the circumcircle of the triangle ABC, PP_1, PP_2, and PP_3 be drawn perpendicular to the respective sides of the triangle, then the points P_1, P_2, P_3 are on the same straight line, which is called Simson's line of P with respect to the triangle ABC. This may be proved as follows: Join P_1P_2 and P_3P_2; then these lines form one straight line. For, join PC and PA, and since angles PP_2C and PP_1C are right angles, the points PP_2CP_1 are concyclic. But since

$BAPC$ and PP_2P_1C are quadrilaterals inscribed in a circle, opposite angles are supplementary, and

$$\measuredangle PAB + \measuredangle PCB = 180°,$$

$$\measuredangle PP_2P_1 + \measuredangle PCP_1 = 180°;$$

whence $\measuredangle PP_2P_1 = \measuredangle PAB.$

Also $\measuredangle P_3AP = \measuredangle P_3P_2P,$

since each is inscribed in the same sequent of the circle.

But $\measuredangle P_3AP + \measuredangle PAB = 180°,$

therefore $\measuredangle P_3P_2P + \measuredangle PAB = 180°.$

That is, $\measuredangle P_3P_2P$ and $\measuredangle PP_2P_1,$

are supplementary adjacent angles, and the points P_1, P_2, P_3 are collinear.

29. If PP_1 is produced until it intersects the circle about the triangle NBC, at the point U, then AU is parallel to Simson's line belonging to P. The points P, P_1, P_2, and C are concyclic, and therefore the angle P_1PC equals angle P_1P_2C. But angle P_1P_2C equals angle UAC, since the arc AU is common to both angles, and $\measuredangle P_1P_2C = \measuredangle UAC$. If two angles are equal and have a pair of sides in coincidence, then the other sides must also either coincide or be parallel. Hence AU is parallel to $P_1P_2P_3$, or to Simson's line.

30. Let AT be a line isogonal conjugate to AP, then Simson's line belonging to P is perpendicular to AT; and the Simson line for T is perpendicular to AP.

We have seen (**29**) that UA is parallel to $P_1P_2P_3$ and that

$$\measuredangle BAT = \measuredangle PAC = \measuredangle PUC$$

also $\measuredangle BAU = \measuredangle BCU;$

by addition

$$\measuredangle BAT + \measuredangle BAU = \measuredangle PUC + \measuredangle BCU.$$

Since $\measuredangle PUC + \measuredangle BCU = 90°$,

therefore $\measuredangle BAT + \measuredangle BAU = 90°$;

or the line AT is perpendicular to AU, and hence also perpendicular to Simson's line $P_1P_2P_3$.

31. The Simson lines belonging to the extremities of a diameter of the circumcircle are perpendicular to each other.

If P and Q, Fig. 9, are the extremities of a diameter, then angle PUQ equals 90° equals PP_1B, and therefore UQ is parallel to BC. Draw QT perpendicular to BC, then PT is parallel to BC, and arc BT is equal to arc PC, and, consequently, $\measuredangle TAB = \measuredangle PAC$. But by a previous proposition (**29**), the Simson line for Q is parallel to AT; and AT is perpendicular to Simson's line belonging to P, or the two Simson lines belonging to the extremities of the diameter PQ are perpendicular to each other.

32. Let P and Q be the points of intersection of Brocard's diameter produced to meet the circumcircle of the triangle ABC; then they will be the extremities of a diameter of the circumcircle itself.

AP and AQ are perpendicular to each other, and therefore their isogonal conjugates will also be perpendicular to each other. As we have seen (**3**), the isogonal conjugates to AP and AQ determine the directions of the points at infinity, or the directions of the asymptotes of the hyperbola isogonal conjugate to Brocard's diameter with respect to the triangle. Since the Simson line belonging to P, Fig. 9, is perpendicular to AT, the isogonal conjugate to AP, and the Simson line to Q is perpendicular to AP, the isogonal conjugate to AQ (**30**); and also the Simson lines belonging to the extremities of a diameter are perpendicular to each other (**31**), therefore the asymptotes of the hyperbola will be parallel to the Simson lines for P and Q. Subsequently we shall prove that the asymptotes of the hyperbola isogonal conjugate to Brocard's diameter are not only parallel to the Simson lines belonging

to the extremities of the prolonged Brocard's diameter on the circumcircle, but that they coincide with them.

33. The orthocentre and the median point of a triangle are respectively the external and internal centres of similitude of the circumcircle and of Feuerbach's circle of the triangle.

In order to find the centres of similitude of the circumcircle and of Feuerbach's circle of a triangle, we must draw parallel radii in both circles. The line joining the extremities of the radii drawn parallel in the same or in the opposite direction will intersect the line joining the centres of the circles in the external or the internal centre of similitude.

Let YY', Fig. 10, be parallel to MX, then must XY pass through H, and XY' through E. The radius of Feuerbach's circle is equal to one-half of the radius of the circumcircle (**26**); that is, FY equals $\frac{1}{2}MX$. By construction, FY is parallel to MX, and FM equals $\frac{1}{2}MH$. Therefore,

$$MX : FY = MH : FH,$$

or X, Y, and H are on the same straight line.

34. If E is the median point, then ME equals $2\,EF$. This is easily established.

Since $\quad MF = \frac{1}{2}MH,$

and $\quad ME = \frac{1}{3}MH,$

therefore $\quad MF - ME = EF = \frac{1}{2}MH - \frac{1}{3}MH = \frac{1}{6}MH,$

and $\quad ME : EF = \frac{1}{3}MH : \frac{1}{6}MH = 2:1,$

or $\quad ME = 2\,EF.$

Hence in the triangles MEX and $EY'F$,

$$MX : FY' = ME : EF,$$

or X, E, and Y' are collinear.

35. The circumcircle and Feuerbach's circle of a triangle divide any line drawn from the orthocentre in the ratio $2:1$.

For, $\quad HX : HY = MH : HF;$

but since $\quad MH : HF = 2:1,$

therefore $\quad HX : HY = 2:1.$

The same will be the case with any line drawn from H to the circumcircle of ABC.

36. The Simson line for P, on the circumcircle of ABC with respect to triangle ABC, bisects the line joining P with the orthocentre of ABC.

Let HP (Fig. 11, Plate 3) cut the Simson line at D; we are to prove that D is a point on Feuerbach's circle.

(Indirect Proof.) Suppose D *is* the middle point of HP, or, what is the same thing, D is a point on Feuerbach's circle; then we have only to prove that the line joining D with the foot of the perpendicular from P to BC, *i.e.* DP_1, is Simson's line.

Since AG and PP_1 are perpendicular to BC, and $HD = DP$, the triangle GDP_1 is isosceles. Let E be the middle point of AH; draw DE. Then DE equals $\frac{1}{2}AP$. The points E, G, and D are on Feuerbach's circle; A, P, and C are on the circumcircle. Since ED equals $\frac{1}{2}AP$, and the radius of Feuerbach's circle equals one-half the radius of the circumcircle, the angle EGD inscribed in Feuerbach's circle will be equal to the angle ACP inscribed in the circumcircle of ABC.

We have then, $\quad \measuredangle EGD = \measuredangle ACP.$

$$\measuredangle EGD = \measuredangle DP_1P.$$

Therefore, $\quad \measuredangle DP_1P = \measuredangle ACP.$

Let the intersection of DP_1 and AC be P_2. Then

$$\measuredangle DP_1P = \measuredangle ACP = \measuredangle P_2CP,$$

or the points P_1, P_2, P, and C are concyclic; therefore,

$$\measuredangle PP_2C = \measuredangle PP_1C = 90°.$$

The point P_2 is therefore the foot of the perpendicular from P to AC; *i.e.* P_2P_1 is Simson's line.

NOTE. — The point D is called the *centre* of Simson's line.

37. The point of intersection of Simson's lines belonging to the extremities of a diameter of the circumcircle of a triangle is on the circumference of Feuerbach's circle for that triangle.

Let W (Fig. 12, Plate 3) be the point of intersection of QH and Simson's line for Q, and let D be the point of intersection

of HP and the Simson line for P; then W and D are middle points of QH and PH respectively, and W and D are on Feuerbach's circle. WD is parallel to and equal to one-half of PQ, and will bisect all lines drawn from H to PQ, and therefore WD will bisect MH at the centre of Feuerbach's circle, F. Hence WD becomes a diameter of Feuerbach's circle, and since angle WZD between the Simson lines is equal to a right angle (**31**), therefore Z is on the circumference of the circle about WD as a diameter; that is, Feuerbach's circle.

NOTE. — This point Z is called the *vertex* of either Simson line. We see then that Feuerbach's circle intersects a Simson line in its *centre* and *vertex* (**36**).

We see then that the asymptotes of the equilateral hyperbola isogonal conjugate to Brocard's diameter must coincide with Simson's lines belonging to the points in which Brocard's diameter intersects the circumcircle of triangle ABC (**32**).

38. In connection with the properties of Simson's line already given, we shall examine some important relations, which shall hereafter be made use of in the chapter on Higher Plane Curves.

39. If P, Fig. 13, be joined with H', H'', and H''', the points of intersection of the altitudes upon the sides produced to meet the circumcircle, and if the points of intersection of of PH', PH'', and PH''' with the three sides be respectively U, V, W; then the points U, V, W, together with the orthocentre H, are collinear, and UVW is parallel to the Simson line to P.

Proof.

$\measuredangle WHH_c = \measuredangle WH'''H_c$ (since Feuerbach's circle bisects HH''' at H_c).

But $\measuredangle WH'''H_c = \measuredangle PH'''C = \measuredangle PBC$ (being in same segment).

Again, $\measuredangle UHH_a = \measuredangle UH'H_a = \measuredangle PH'A = \measuredangle PBA$; hence, by addition,

$$\measuredangle WHH_c + \measuredangle UHH_a = \measuredangle PBC + \measuredangle PBA = \measuredangle ABC.$$

Adding $\measuredangle H_aHH_c$ to both sides, we have,

$$\begin{aligned}\measuredangle WHH_c + \measuredangle UHH_a + \measuredangle H_aHH_c &= \measuredangle ABC + \measuredangle H_aHH_c \\ &= \measuredangle H_cBH_a + \measuredangle H_aHH_c \\ &= \measuredangle 180^\circ.\end{aligned}$$

Therefore, W, H, and U are collinear; and in like manner it may be shown that W, H, and V are collinear.

40. If PP_2 is produced to meet the line UVW at X, then PX is parallel to HH'', since both are perpendicular to AC. The portion $H''H_b$ equals the portion H_bH (**35**), since Feuerbach's circle bisects any line from the orthocentre; therefore, $PP_2 = P_2X$.

Also, $\measuredangle P_2XV = \measuredangle P_2PV = \measuredangle H_bH''V = \measuredangle PCP_1$; and the figure $H''PCB$ being an inscribed quadrilateral, the points P_2, P_1, C, and P are concyclic, and

$$\measuredangle PP_2P_3 \text{ or } \measuredangle P_2XV = PCP_1 = \measuredangle H_bH''V.$$

Wherefore, the line WUV is parallel to Simson's line for P.

41. All straight lines drawn from P to WUV are bisected by the Simson line for P. Hence, as a special case, PH is bisected. This is a simpler proof of the same theorem given some pages back (**36**).

We shall call D, the point in which the Simson line is intersected by the line joining P and the orthocentre, H, the centre of Simson's line (**36**).

42. The angle between two Simson's lines belonging to two points P and P' on the circumcircle of ABC, is equal to the inscribed angle upon the arc between P and P' on the circle about ABC, and also to the angle in Feuerbach's circle upon the arc between the centres of the Simson lines.

Let the line joining P' to H' (Fig 13) intersect BC at U', then HU' is parallel to the Simson line belonging to P'. Since $HU = H'U$, also $HU' = H'U'$, and UU' is common, hence $\triangle HU'U = \triangle H'U'U$, and $\measuredangle UHU' = \measuredangle UH'U' = \measuredangle PH'P'$. But since HU and HU' are parallel to the Simson lines for P and

P', the angle between these Simson lines is equal to the angle between PH' and $P'H'$ or $= \measuredangle PH'P'$.

We have already shown that H is the centre of similitude of the circumcircle of ABC, and the Feuerbach circle (**33**); therefore P' and D', P and D, H' and H_a, are corresponding points in the circles. It follows that H_aD, $H'P$, H_aD', and $H'P'$ are corresponding lines, and $\measuredangle DH_aD' = \measuredangle PH'P'$, or the inscribed angle in Feuerbach's circle upon the arc between the centres of the Simson lines, is equal to the angle between the Simson lines.

We have seen that the Simson lines corresponding to the extremities of a diameter are perpendicular to each other, and they intersect on the circumference of Feuerbach's circle (**31**, **37**). This point of intersection of both lines and the circle we have called the *vertex* of either Simson line (**37**, **N.**). Feuerbach's circle, therefore, passes through both vertex and centre of the Simson line (**37**, **N.**).

43. A side, BC, and its altitude in a triangle are the Simson lines corresponding to the extremities of the diameter AA' (Fig. 16).

For, since the feet of the perpendiculars from A to AB and AC coincide with A, and the foot of the perpendicular from A to BC is H_a, therefore the Simson line for A is AH_a. Again, the feet of the perpendiculars from A' to the sides AB, AC, BC of the triangle are B, C, and A'_a, respectively; hence BC is the Simson line for A'.

44. If through an angular point A of a triangle ABC, Fig. 16, the diameter of the circumcircle be drawn to meet the circle at A', then the line joining A' with the orthocentre H will bisect the side BC.

This follows easily from known properties. For, the Simson line to A' is BC (**43**), therefore BC must bisect $A'H$, and, as we have shown, this bisection is on Feuerbach's circle (**36**). But Feuerbach's circle cuts BC at two places only, that is at H_a and M_a; hence it is obvious that $A'H$ passes through M_a, and the proposition is established.

45. Simson's lines corresponding to the extremities of a diameter are said to be conjugate.

46. The points of intersection, E and F, Fig. 16, of a Simson line, with a pair of conjugate Simson lines, are equally distant from the centre T' on $T'S'$, and $EF = 2\,T'S$.

Let $T'S'$ be the Simson line of a point on the circumcircle of ABC, and cutting the conjugate Simson lines DS and TS at E and F, then $\measuredangle\, T'ED = \measuredangle\, T'SD$, since the angle between two Simson lines equals the inscribed angle in Feuerbach's circle on the arc between the centres. Therefore the triangle $ET'S$ is isosceles, and $\measuredangle\, T'ES = \measuredangle\, T'SE$, whence ET' equals $T'S$. Since $\measuredangle\, ESF = 90°$, $\measuredangle\, T'SF = \measuredangle\, T'FS$, and $T'S = T'F$; or $ET' = T'F = T'S$.

47. The arc between the vertices of two Simson lines (not conjugate) is twice as large as the arc between their centres. For $ET'S$ is an isosceles triangle, and

$$\measuredangle\, FT'S = \measuredangle\, S'T'S = 2\,T'SD,$$

or the arcs measuring these angles are in the ratio of $1:2$.

48. If $T'E = T'F = T'S$ is less than the diameter of Feuerbach's circle, or less than the radius of the circumcircle of triangle ABC, then a circle from T' as centre and with $T'E$ as radius will cut Feuerbach's circle a second time, at S'', and since $T'S'' = T'E = T'F$, S'' will be the vertex of another pair of conjugate Simson's lines, which will pass through E and F. It is evident that there are always two pairs of conjugate Simson lines passing through E and F.

49. If $T'S = T'S''$ diminishes, the points S and S'' will approach the position of T', but from opposite directions. If $T'S = T'S''$ becomes zero, then S, S'', E, and F coincide with T'. Then the angle between the vertices, or $\measuredangle\, ST'S'' = 180°$, whence the angle between the centres is $90°$ (**47**), or, the two Simson lines passing through T' will be perpendicular to each other. These lines are $T'S'$ and a perpendicular to it at T', T' being the common centre.

50. If $T'S$ becomes equal to the diameter of Feuerbach's circle, S and S'' will coincide at S''', and $T'S'''=T'E=T'F=r$. In this case we have only one pair of conjugate Simson lines, $S'''L$ and $S'''L'$.

Since one pair of Simson lines (conjugate) intersect on Feuerbach's circle, therefore $T'S'''=T'L=T'L'$ will not become greater than the diameter of Feuerbach's circle, and therefore no Simson line will intersect LL' beyond the points L and L'; and through all points between LL', two pairs of conjugate Simson lines may be drawn, except for the cases where S and S'' coincide with either T' or S'''. We shall call the points L and L' the *limiting points*, and $S'''L$ and $S'''L'$ the *limiting Simpson lines* with respect to $T'S'$. Also the limiting points T and T' will be referred to as the *points of contact* of the limiting Simson lines.

We have now a method for the construction of any number of conjugate Simson lines, if one Simson line corresponding to a point on the circumcircle of a triangle be given. If the Simson line $T'S'$ and Feuerbach's circle are given (Fig. 16), we have only to join any point S on the Feuerbach's circle to T' and to take $T'E=T'F=T'S$. Then SE and SF are a pair of conjugate Simson lines.

51. If two Simson lines SD and $S''D''$ which are not conjugate cut a third $T'S'$ at equal distances, E and F, from its centre T', then the line joining the point of intersection, K, of SD and $S''D''$ with S', the vertex of $T'S'$, is a Simson line conjugate to $T'S'$.

Let ES and FS'' intersect at K, and ES'' and FS intersect at N. Since the pairs of lines ES, FS and ES'', FS'' are conjugate Simson lines, they are perpendicular to each other; or ES'' and FS are altitudes in the triangle EKF. Therefore, KN is the third altitude, and we may prove S' to be the foot of this altitude on EF.

Feuerbach's circle of the triangle ABC passing through the feet S and S'' of the two altitudes, and through the middle point T' of one side of the triangle EKF, must be also Feuer-

bach's circle of the triangle EKF, and therefore the second intersection of Feuerbach's circle with the side EF must be the foot of the altitude to EF, or KS'. Hence S' is the foot of the altitude KN. But, as we have seen before (**43**), any side and its altitude is a pair of conjugate Simson lines, and since EF is a Simson line of a point on the circumcircle of ABC, with respect to the triangle, KN is the Simson line conjugate to EF.

52. Any triangle like EKF formed by three Simson lines, the altitudes of which are Simson lines conjugate to the sides, and having Feuerbach's circle in common with the triangle ABC, we shall call *Simson's Triangle*.

53. Since Feuerbach's circle is common to both triangles ABC and EKF, the radius of Feuerbach's circle is one-half the radius of the circumcircle of either triangle; therefore the radius of the circumcircle of any Simson triangle is equal to the radius of the circumcircle of the original triangle.

54. The common vertex S''' of the pair of limiting Simson lines belonging to $T'S'$ is on the same straight line as K, N, and S'.

For, since $T'S'''$ is a diameter of Feuerbach's circle, $\measuredangle\, T'S'S''' = 90°$, or $S'S'''$ is perpendicular to EF; therefore S''' is on the altitude to EF in the triangle EKF.

55. The point of contact L and the vertex S''' of a limiting Simson line, with respect to $T'S'$, are equally distant from the centre of the line; or $RL = RS'''$.

Let $S'''L$ intersect Feuerbach's circle in R, then $\measuredangle\, T'RS''' = 90°$. But $\measuredangle\, L'S'''L = 90°$, hence $T'R$ is parallel to $L'S'''$, and since $L'T' = T'L$, $RS''' = RL$. In like manner it may be proved that R' is the middle point between L' and S'''.

2. THE ELLIPSE.

56. The conic isogonal conjugate to Lemoine's line, with respect to the triangle, is an ellipse, called Steiner's ellipse, whose centre is the median point of the triangle. Lemoine's line is the polar to Grebe's point with respect to the circumcircle of the triangle.

To the point A, Fig. 17, as a pole with respect to the circumcircle of ABC belongs the polar AK''_a which is the tangent to the circle at A. To P_a, which is the point of intersection of the tangents drawn at B and C, and at the same time a point on the Symmedian line through A, belongs the polar BC. The intersection of the two given polars is the pole of the line joining the two respective poles of the given polars. Therefore K''_a is the pole to AP_a, or to the Symmedian line through A. If K'_a is the point of intersection of the Symmedian AK and the circle about ABC, then $K''_aK'_a$ is a tangent to the circle at the point K'_a. Similarly, K''_b is the pole to the Symmedian BK_b, and K''_c is the pole to the Symmedian CK_c. The points K''_a, K''_b, K''_c, are the poles to the three straight lines AK_a, BK_b, CK_c, and since these are concurrent, the point of concurrency being Grebe's point K, therefore the points K''_a, K''_b, and K''_c, will be collinear. This line is called Lemoine's line with respect to the triangle ABC. AK_a and BK_b intersect in K. A, B, K'_a and K'_b are concyclic; but since AK'_a and BK'_b intersect in K, therefore AK'_b and BK'_a will intersect at a point Q_c on the polar to K, *i.e.* to Lemoine's line; and CK passes through Q_c.

57. If the median line CM_c be produced, and M_cE''' be made equal to M_cE, then AK'_b or AQ_c is isogonal conjugate to AE''', and BK'_a or BQ_c is isogonal conjugate to BE'''. Hence the intersection of AQ_c and BQ_c or Q_c is isogonal conjugate to the intersection of AE''' and BE''', *i.e.* the point E'''. This may be proved as follows:

Since M_c is the middle point of AB, and $M_cE''' = M_cE$, therefore $AE'''BE$ is a parallelogram, and $\measuredangle BAE''' = \measuredangle ABC$.

But since BM_b and BK_b are isogonal conjugates, therefore $\measuredangle ABE = \measuredangle KBC = \measuredangle K'_bBC$. But $\measuredangle K'_bBC = \measuredangle K'_bAC$, therefore $\measuredangle BAE''' = \measuredangle CAK'_b$, and AK'_b or AQ_c is isogonal conjugate to AE'''. In the same way it can be proved that the $\measuredangle ABE''' = \measuredangle CBK'_b$, and BK'_a or BQ_c is isogonal conjugate to BE'''. Likewise, Q_a is isogonal conjugate to E', and Q_b to E''.

58. With respect to the triangle ABC, the ellipse isogonal conjugate to Lemoine's line, and passing through E', E'', E''', will also pass through the angular points A, B, and C of the triangle (**2**). The isogonal conjugate to BK''_a is BA, and the isogonal conjugate to CK''_a is CA; K''_a is, therefore, a point isogonal conjugate to A. In the same way, we may show K''_b to be isogonal conjugate to B, and K''_c to C. The ellipse will therefore pass through A, B, C, E', E'', and E'''. Since $EE' = AE$, $EE'' = BE$, and $EE''' = CE$, the point E will be the centre of the ellipse.

59. One of the points of intersection of the ellipse and the circumcircle of the triangle is R, which is called Steiner's point. (See Fig. 15.) Since R is on the circumcircle of ABC, its isogonal conjugate point is at infinity; and it is on Lemoine's line, as will now be proved. A line drawn through A, parallel to Lemoine's line, will be isogonal conjugate to AR.

We shall first examine some properties of the point R. The lines drawn through the angular points of the triangle ABC parallel to the respective sides of Brocard's triangle, or, what is the same thing, perpendicular to AN, BN, and CN (**18**) (N being Tarry's point), intersect in one point R, which is called Steiner's point, and is on the circle about ABC.

60. Let the lines drawn at A perpendicular to AN, and at B perpendicular to BN, intersect at R. Since $\measuredangle NAR = 90°$, and $\measuredangle NBR = 90°$, the points A, N, B, R are concyclic, and R must be on the circumcircle of ANB; and A, N, and B are

on the circumcircle of ABC, whence R is on circumcircle of ABC. Since $\measuredangle NAR = 90°$, NR is a diameter of this circle (N, M, and R are collinear), and $\measuredangle NCR = 90°$, *i.e.* the perpendicular to CN at C passes through R. We are now to prove that R is isogonal conjugate to the point at infinity in the direction of Lemoine's line. Since Lemoine's line is the polar to K, MK will be perpendicular to Lemoine's line. The ray drawn through A to the point at infinity on Lemoine's line is parallel to this line and is perpendicular to MK.

Let this perpendicular cut the circumcircle of ABC at R' (Fig. 17), then $\measuredangle R'AB = \measuredangle CAR$, *i.e.* AR is isogonal conjugate to AR'. Since AR is parallel to the side B_1C_1 of Brocard's triangle, the angle between AR and BC is the same as the angle between B_1C_1 and BC. But the angle between B_1C_1 and BC is equal to the angle AMK, which we may denote by ϕ_a (**24**). The angle at the centre, M, over the arc $AR' = 2\phi_a$, hence the inscribed angle over the arc AR' equals ϕ_a. Let the point where AR cuts the side BC of the triangle ABC be R_a, then $\measuredangle AR_aB = \phi_a$. We have then

$$\measuredangle CAR_a = \measuredangle CAR = \measuredangle AR_aB - \measuredangle ACB = \phi_a - \gamma,$$

in which γ represents $\measuredangle C$ of triangle ABC.

$$\text{Arc } R'B = \text{arc } AB - \text{arc } AR' = \gamma - \phi_a,$$

hence $\measuredangle R'AB = \measuredangle CAR$, or AR is isogonal conjugate to AR', or to the line parallel to Lemoine's line. In the same way it may be shown that BR is isogonal conjugate to BR'', which is parallel to Lemoine's line through B; and that CR is isogonal conjugate to CR''', which is parallel to Lemoine's line through C. The point R is, therefore, the point isogonal conjugate to the point on Lemoine's line at infinity.

61. If a circle and an ellipse intersect, the directions of the axes of the ellipse may be found by bisecting the angles between the common chords of the ellipse and the circle. Thus, as A, B, C, and R are points common to both circle and ellipse, the directions of the axes of the ellipse will be given by the two bisectors of the angles between the common chords

AR and BC. Since AR is parallel to B_1C_1, the line through the median point E and parallel to the bisector of the angle between BC and B_1C_1 will be one of the axes; and a line also through E, and perpendicular to this, will be the other axis.

62. The axes of the ellipse isogonal conjugate to Lemoine's line with respect to the triangle are parallel to the asymptotes of the hyperbola isogonal conjugate to Brocard's diameter. (**37**). This may be proved as follows:

If upon the sides of the triangle ABC, Fig. 18, are constructed similar isosceles triangles, BA_2C, CB_2A, and AC_2B, then the triangles ABC and $A_2B_2C_2$ have the same median point E. The proof is similar to that in the special case when the vertices of the similar isosceles triangles are the angular points of Brocard's triangle (**10**). Let A_2, B_2, and C_2 be the vertices of these similar isosceles triangles constructed upon the sides of ABC, and let KA_2, KB_2, and KC_2 meet the sides BC, AC, and AB respectively at $A_{2\alpha}$, $B_{2\beta}$, and $C_{2\gamma}$, then it can be proved that triangle $A_{2\alpha}B_{2\beta}C_{2\gamma}$ is similar to triangle $A_2B_2C_2$, the centre of similitude being K. If we erect a perpendicular at $A_{2\alpha}$ to BC to meet Brocard's diameter at Q_2, then, putting for A_1M_a, B_1M_b, their equals, KK_a, KK_b respectively, we have

$$\frac{A_2M_a}{KK_a}=\frac{A_{2\alpha}A_2}{A_{2\alpha}K}=\frac{Q_2M}{Q_2K}.$$

Since the triangles A_2BC and B_2AC are similar, we have

$$\frac{A_2M_a}{B_2M_b}=\frac{M_aC}{M_bC}=\frac{a}{b}=\frac{A_1M_a}{B_1M_b},$$

or
$$\frac{A_2M_a}{A_1M_a}=\frac{B_2M_b}{B_1M_b}=\frac{B_{2\beta}B_2}{B_{2\beta}K}=\frac{Q_2M}{Q_2K}.$$

Therefore
$$\frac{A_{2\alpha}A_2}{A_{2\alpha}K}=\frac{B_{2\beta}B_2}{B_{2\beta}K}.$$

Similarly, we get
$$\frac{B_{2\beta}B_2}{B_{2\beta}K}=\frac{C_{2\gamma}C_2}{C_{2\gamma}K}=\frac{Q_2M}{Q_2K},$$

or, triangles $A_2B_2C_2$ and $A_{2\alpha}B_{2\beta}C_{2\gamma}$ are similar, and K is the centre of similitude.

From the equation $\frac{B_{2\beta}Q_2}{B_{2\beta}K} = \frac{Q_2M}{Q_2K}$, it follows that $B_{2\beta}Q_2$ is parallel to B_2M, and since B_2M is perpendicular to AC, therefore $B_{2\beta}Q_2$ is also perpendicular to AC, or the perpendicular at $B_{2\beta}$ to AC passes through Q_2. Similarly, the perpendicular at $C_{2\gamma}$ to AB passes through Q_2. If, now, Q_2 is caused to coincide with either Q_3 or Q_4, the points of intersection of Brocard's diameter and the circumcircle of the triangle ABC, then the triangle $A_{2a}B_{2\beta}C_{2\gamma}$ will degenerate into the straight lines $Q_{3a}Q_{3b}Q_{3c}$ and $Q_{4a}Q_{4b}Q_{4c}$, which are the Simson lines belonging to Q_3 and Q_4 with respect to the circumcircle of the triangle. The triangle $A_2B_2C_2$ will degenerate into the straight lines $A_3B_3C_3$ and $A_4B_4C_4$, which will be parallel to the Simson lines belonging to Q_3 and Q_4; and they will pass through the median point E, for the lines $A_3B_3C_3$ and $A_4B_4C_4$ still have the median point E in common with ABC. Also, A_3, B_3, C_3 and A_4, B_4, C_4 are on the perpendiculars at the middle points of the respective sides of the triangle ABC. Since the Simson lines to Q_3 and Q_4 correspond to the extremities of a diameter, they are perpendicular to each other, and therefore their parallels $A_3B_3C_3$ and $A_4B_4C_4$ are also perpendicular to each other.

Furthermore, $$Q_{3a}M_a = Q_{4a}M_a,$$

$$Q_{3a}M_a : M_aK_a = Q_{4a}M_a : M_aK_a,$$

$$Q_{3a}M_a : M_aK_a = Q_{3a}A_3 : A_3K = A_3M_a : A_3A_1,$$

and $$Q_{4a}M_a : M_aK_a = Q_{4a}A_4 : A_4K = A_4M_a : A_4A_1,$$

or $$A_3M_a : A_3A_1 = A_4M_a : A_4A_1,$$

whence $\{M_aA_1, A_3A_4\}$ is an harmonic range, and $E\{M_aA_1, A_3A_4\}$ is an harmonic pencil. Since $\measuredangle A_4EA_3 = 90°$, EA_3 will bisect the angle A_1EM_a.

63. Now, in two similar triangles the bisectors of the angles formed by any line in one triangle with the corresponding line in the other triangle are parallel to each other, hence the bisector of the angle formed by A_1E and EM_a, or the line AE,

i.e. the line $A_3B_3C_3$, is parallel to the bisector of the angle formed by B_1C_1 and BC. But the bisector of the angle formed by B_1C_1 and BC is parallel to one axis of the ellipse whose centre is at E, therefore the line $A_3B_3C_3$ coincides with one axis, and $A_4B_4C_4$ coincides with the second axis of the ellipse. We have seen (**62**), however, that $A_3B_3C_3$ and $A_4B_4C_4$ are parallel to Simson's lines belonging to the extremities of Brocard's diameter, which are themselves parallel to the asymptotes of the equilateral hyperbola isogonal conjugate to Brocard's diameter; hence the axes of the ellipse isogonal conjugate to Lemoine's line are parallel to the asymptote of the equilateral hyperbola isogonal conjugate to Brocard's diameter.

64. Before taking up other properties of the ellipse isogonal conjugate to Lemoine's line with respect to a triangle, we shall examine the properties of the fifth remarkable point and its relations to the other points in the triangle.

65. It may be of interest to state at this point that O and O' (Fig. 15) are the foci of an ellipse inscribed in the triangle ABC. The points of contact of the sides with the inscribed ellipse are K_α, K_β, and K_γ (the points of intersection of the Symmedian lines with the sides of the triangle). We know that the focal radii to the point of contact of a tangent to an ellipse form equal angles with the tangent.

Since
$$\frac{BK_a}{CK_a}=\frac{c^2}{b^2},$$

and
$$BO=\frac{ac^2}{\sqrt{a^2b^2+a^2c^2+b^2c^2}},\quad CO'=\frac{ab^2}{\sqrt{a^2b^2+a^2c^2+b^2c^2}},\qquad \textbf{(14)}$$

therefore
$$\frac{BO}{CO'}=\frac{c^2}{b^2}=\frac{BK_a}{CK_a};$$

and, as $\measuredangle\, OBC=\measuredangle\, O'CB=\delta$, therefore triangles OK_aB and $O'K_aC$ are similar. This proves that $\measuredangle\, OK_aB=\measuredangle\, O'K_aC$; and O and O' are the foci, and K_a is the point of contact of the side BC with the inscribed ellipse.

66. The lines drawn from the vertices of a triangle to the points of contact of the three enscribed circles with the sides of the triangle are concurrent.

Let O_a, O_b, and O_c (Fig. 19) be the points of contact of the inscribed circle with the respective sides of the triangle, and Q'_a, Q'_b, Q'_c the points of contact of the three escribed circles; then

$$CO_a = BQ'_a,\ CO_b = AQ'_b,\ AO_c = BQ'_c, \text{ etc.}$$

(Any side of a triangle is divided by the points of contact of the inscribed and escribed circles into three parts, the two extreme parts of which are equal.)

For, $\qquad CO_a = BQ'_a,\ CO_b = AQ'_b;$

but $\qquad CO_a = CO_b$, hence $BQ'_a = AQ'_b$. (1)

Again, $\qquad AO_b = CQ'_b,\ AO_c = BQ'_c;$

but $\qquad AO_b = AO_c$, hence $CQ'_c = BQ'_c$. (2)

Also, $\qquad BO_a = CQ'_a,\ BO_c = CQ'_c;$

but $\qquad BO_a = BO_c$, hence $CQ'_a = CQ'_c$. (3)

Multiplying these results, we have

$$AQ'_b \cdot CQ'_a \cdot BQ'_c = AQ'_c \cdot BQ'_a \cdot CQ'_b,$$

and from Ceva's proposition it follows that AQ'_a, BQ'_b, CQ'_c are concurrent.

The point Q is called the fifth remarkable point, or Nagel's point of the triangle.

67. The line joining an angular point of the triangle with the point of contact of the escribed circle belonging to the opposite side is parallel to the line joining the middle point of this side with the centre of the inscribed circle (Fig. 19).

We are to prove that OM_a is parallel to AQ'_a. The point A is the centre of similitude of the inscribed and escribed circles. Let OO_a, produced, meet AQ'_a at the point N, then

$$AO' : AO = O'Q'_a : OO_a,$$

and $\qquad AO' : AO = O'Q'_a : ON,$

whence $\qquad OO_a = ON.$

Therefore $O_aM_a = M_aQ'_a$, and OM_a is parallel to AQ'_a.

We shall now prove that AQ equals twice OM_a, BQ equals twice OM_b, and CQ equals twice OM_c. Since OM_a is parallel to AQ, and OM_b is parallel to BQ, and M_aM_b is parallel to AB, therefore the triangles OM_aM_b and AQB are similar, whence

$$AQ : OM_a = AB : M_aM_b = 2 : 1,$$

or AQ equals twice OM_a.

68. The fifth remarkable point, the median point, and the centre of the inscribed circle of a triangle are collinear; and the median point divides the distance between the fifth remarkable point and the centre of the inscribed circle in the ratio of $2 : 1$.

Join E with Q and O;

then $$AQ : OM_a = 2 : 1, \tag{67}$$

and $$AE : EM_a = 2 : 1; \tag{10}$$

also OM_a is parallel to AQ,

therefore $$\measuredangle QAM_a = \measuredangle EM_aO.$$

From this it follows that the triangles AQE and EOM_a are similar, therefore

$$\measuredangle OEM_a = \measuredangle AEQ,$$

and the points Q, E, O are collinear. From the similar triangles AQE, EOM_a it also follows that

$$AE : AM_a = QE : OE = 2 : 1,$$

or $$QE = 2\,OE.$$

69. It is interesting to notice the analogy between Q, E, O, and H, E, M.

Since $$QE : EO = 2 : 1,$$

and $$HE : EM = 2 : 1,$$

H, Q, M, and O are the angular points of a trapezoid, and the median point of the triangle is the point of intersection of the diagonals of this trapezoid.

70. The middle point P, of QO, is the centre of the circle inscribed in the triangle $M_aM_bM_c$.

$$PO = \tfrac{1}{2}\,OQ,$$

$$OE = \tfrac{1}{3}\,OQ,$$

hence $$PO - OE = PE = \tfrac{1}{6}\,OQ,$$

therefore $$OE : EP = 2 : 1.$$

But since $$AE : EM_a = 2 : 1, \qquad (\mathbf{10})$$

the triangle AEO is similar to the triangle M_aEP, and AO is parallel to M_aP.

We know that M_aM_c is parallel to AC,

therefore $$\measuredangle\, OAM_b = \measuredangle\, PM_aM_c,$$

and $$\measuredangle\, OAM_c = \measuredangle\, PM_aM_b.$$

But since $$\measuredangle\, OAM_b = \measuredangle\, OAM_c,$$

therefore $$\measuredangle\, PM_aM_c = \measuredangle\, PM_aM_b;$$

that is, the line PM_a bisects the angle $M_cM_aM_b$. In like manner it may be proved that PM_b bisects angle $M_aM_bM_c$. Hence P is the centre of the circle inscribed in the triangle $M_aM_bM_c$.

71. The centre of Feuerbach's circle circumscribing $M_aM_bM_c$ is F. We notice that P is the middle point of OQ, and F is the middle point of HM; and, indeed, the analogy between Q and H may be noticed throughout. F is on the line joining the middle point of BC with the middle point of the upper portion of the altitude drawn to BC (**36**); and P is on the line joining the middle point of BC with the middle point of the upper part of AQ'_a.

This last may be proved as follows:

Let PM_a (Fig. 19, Plate 3) meet AQ at the point R; then since PO equals PQ, and OM_a is parallel to AQ, the triangle POM_a is equal to the triangle PQR, or OM_a equals QR. But PM_a is parallel to AO, and OM_a is parallel to the line AR,

therefore it follows that the figure AOM_aR is a parallelogram and $OM_a = AR$. Then OM_a becomes equal to $QR = AR$, and R is the middle point of AQ.

72. Let A', B', and C' (Fig. 20) be the middle points of the arcs subtended by the sides BC, AC, and AB of the triangle ABC; and let $A'M_a$ be produced to meet the circumference a second time in A''', and on this line, which is a diameter of the circle, take A'' so that M_aA'' shall equal M_aA'; then the figure $AHA''A'''$ is a parallelogram, and AA' is perpendicular to AA'''.

$$A''A''' = A'''M_a - A''M_a;$$

but $\quad A'''M_a = r + MM_a,$ (r being the radius),

and $\quad A''M_a = A'M_a = r - MM_a;$

hence $\quad A''A''' = r + MM_a - r + MM_a = 2\,MM_a.$

But the upper part of the altitude drawn to any side is twice the length of perpendicular from the centre of the circumcircle to the same side of the triangle,

therefore $\quad AH = 2\,MM_a,$ and $A''A''' = AH.$

$A''A'''$ is also parallel to AH, since they are perpendicular to the same side BC, and the figure $AHA''A'''$ is a parallelogram. In the same way it may be shown that $BHB''B'''$ is a parallelogram, and that $CHC''C'''$ is a parallelogram.

The angle $A'''AA'$ is a right angle, or $A'''A$ is perpendicular to AA'. But $A'''A$ is parallel to $A''H$, and $A''H$ is therefore perpendicular to AA'. Likewise $B''H$ is perpendicular to BB', and $C''H$ to CC'.

73. Locate Q, the fifth remarkable point, and the points A'', B'', C'', Q, and H are concyclic. This may be proved thus:

AQ is parallel to and equal to twice OM_a (**67**). If on AO prolonged, OQ_a be made equal to AO, and Q_a be joined with Q, then the points Q_a, M_a, and Q are collinear, and M_aQ_a equals M_aQ; and since $A''M_a$ equals M_aA', figure $A''QA'Q_a$ is a parallelogram. QA'' is parallel to $A'Q_a$, or parallel to AA'. Since AA' is perpendicular to $A''H$, QA'' is also perpendicular to

HA''. In a manner analogous to this we may find that $B''Q$ is perpendicular to HB'', and $C''Q$ is perpendicular to HC''. The length HQ is, therefore, the diameter of the circle passing through A'', B'', and C''.

Since $\quad \measuredangle AOC' = \measuredangle AOC + \measuredangle ACO,$

or $\quad \measuredangle AOC' = \measuredangle \tfrac{1}{2} A + \measuredangle \tfrac{1}{2} C;$

and
$$\begin{aligned} \measuredangle OAC' &= \measuredangle OAB + \measuredangle BAC', \\ &= \measuredangle OAB + \measuredangle BCC' \\ &= \measuredangle \tfrac{1}{2} A + \measuredangle \tfrac{1}{2} C; \end{aligned}$$

therefore $\quad \measuredangle AOC' = \measuredangle OAC',$

or $\quad C'A = C'O$, and $B'A = B'O$.

$B'C'$ is, then, perpendicular to AO at its middle point, or $A'O$ is perpendicular to $B'C'$. By similar reasoning $B'O$ may be shown to be perpendicular to $A'C'$, and $C'O$ to $A'B'$. The point O is, therefore, the orthocentre of the triangle $A'B'C'$.

74. HA'' is perpendicular to AO, and $B'C'$ is also perpendicular to AO; hence HA'' is parallel to $B'C'$, and HB'' is parallel to $A'C'$. Then the angle $B''HA''$ is equal to the angle $A'C'B'$. But H and C' are on the circumference of the same circle, and

$$\measuredangle B''HA'' = \measuredangle B''C''A'',$$

or $\quad \measuredangle C' = \measuredangle C''.$

Likewise $\quad \measuredangle B' = \measuredangle B'',$

and $\quad \measuredangle A' = \measuredangle A''.$

The triangles $A''B''C''$ and $A'B'C'$ are therefore similar.

75. If M' is the centre of the circle about the triangle $A''B''C''$, then $M'OMQ$ is a parallelogram. For the triangle $M_aM_bM_c$ is similar to ABC; and since M_aO is parallel to CQ, M_bO parallel to BQ, and M_cO parallel to CQ, therefore the corresponding point to Q in the triangle ABC is the point O in the triangle $M_aM_bM_c$. Hence O is the fifth remarkable point in triangle $M_aM_bM_c$.

The line joining H and Q in triangle ABC will be homologous to the line OM in the triangle $M_aM_bM_c$. In these triangles all homologous parts except the angles are in the ratio $2:1$, therefore HQ equals $2\,OM$. Since M' is the middle point of HQ, therefore OM will be equal to $M'Q$. But OM is parallel to $M'Q$, and therefore $M'OMQ$ is a parallelogram.

76. The perpendiculars from the angular points of ABC to the respective sides of $A''B''C''$ (Fig. 20) intersect in one point, R.

Let R be the intersection of the perpendiculars from A and B to $C''B''$ and $A''C''$ respectively. The perpendicular from C to $A''B''$ also passes through R. For, let γ represent angle C of triangle ABC, $\measuredangle A''C''B'' = 90 - \frac{\gamma}{2}$, hence $\measuredangle ARB$ will be either $90 - \frac{\gamma}{2}$ or $90 + \frac{\gamma}{2}$, $\left(\text{in this case it is } 90 + \frac{\gamma}{2}\right)$.

$$\measuredangle AC''B = \measuredangle AC'B = 180 - \gamma,$$

hence C'' is centre of the circle passing through A, R, and B. The perpendicular from C'' to AR, or the line $B''C''$, bisects AR; and the perpendicular from C'' to BR, or the line $A''C''$, bisects BR. Hence A'' is equally distant from B and R, and also from B and C. Similarly, B'' is equally distant from R and C. CR therefore must pass through R and be perpendicular to $A''B''$.

77. The points of intersection of the corresponding sides of the triangles $A'B'C'$ and $A''B''C''$ are collinear, and the line joining these points is perpendicular to and bisects OR (Fig. 20).

Let $A'B'$ and $A''B''$ intersect at W. $A'B'$ is perpendicular to and bisects CO, therefore every point on $A'B'$ is equally distant from C and O. Every point on $A''B''$ is equally distant from C and R. Hence W, the intersection of $A'B'$ and $A''B''$, is equally distant from C and R. Similarly, we find that U, the intersection of $B'C'$ and $B''C''$, and V, the intersection of $A'C'$ and $A''C''$, are equally distant from O and R; or WVU is a straight line perpendicular to and bisecting OR.

78. If a distance equal to $2r$, the diameter of the inscribed circle of the triangle ABC, be laid off from the vertices on each of the altitudes of the triangle ABC, we obtain three points, A_4, B_4, C_4, which are the vertices of a new triangle; and this triangle is concyclic with triangle $A''B''C''$, and is similar to triangle ABC.

In the triangles $M_aM_bM_c$ and ABC any two corresponding lines are in the ratio $1:2$. O, which is the fifth remarkable point of triangle $M_aM_bM_c$, corresponds to Q in the triangle ABC. OM_a is therefore equal to one-half AQ, and the perpendicular OO_a from O to BC is one-half the perpendicular QN_a drawn from Q to the line through A parallel to the side BC. Also this perpendicular QN_a is equal to $2r$, the diameter of the inscribed circle of ABC. Since AA_4 also equals $2r$, QA_4 is parallel to BC, or $\measuredangle AA_4Q = \measuredangle HA_4Q = 90°$; therefore A_4 is on the circumcircle of HQ. The same is true of B_4 and C_4. The points A_4, B_4, C_4 are then concyclic with $A''B''C''$.

79. Since QA_4 is parallel to BC, and QB_4 is parallel to AC (**78**), $\measuredangle A_4QB_4 = \measuredangle ACB$. But $\measuredangle A_4QB_4 = \measuredangle A_4C_4B_4$, and therefore $= ACB$. Similarly $\measuredangle A_4B_4C_4 = \measuredangle ABC$, and $\measuredangle B_4A_4C_4 = \measuredangle BAC$. Triangle $A_4B_4C_4$ is therefore similar to the triangle ABC.

80. O is the centre of the in-circle of $A_4B_4C_4$ and at the same time the orthocentre of $A''B''C''$. For, $AA_4 = 2r$, and $A''A' = 2M_aA'$; hence

$$AA_4 : A'A'' = r : M_aA'.$$

The triangles AOO_a and M_aBA' are similar ($\measuredangle OAO_a = \measuredangle M_aBA' = \frac{1}{2}\alpha$), hence $AO : A'B = OC : M_aA'$. But $OC = r$, and $A'B = A'O$; hence

$$AO : A'O = r : M_aA',$$

or

$$AO : A'O = AA_4 : A'A''.$$

From the last proposition it follows that the triangles AA_4O and $A'A''O$ are similar.

and $AO : A'O = A_4O : A''O.$

Likewise $BO : B'O = B_4O : B''O,$

and also $CO : C'O = C_4O : C''O.$

There is the same relation between the triangles $A_4B_4C_4$ (similar to ABC) and $A''B''C''$ (similar to $A'B'C'$) with respect to the point O, as between the triangles ABC and $A'B'C'$; that is, O is the centre of the in-circle of $A_4B_4C_4$, and at the same time the orthocentre of $A''B''C''$.

81. $A''B''$ is perpendicular to and bisects OC_4; $A''C''$ is perpendicular to and bisects OB_4; and $B''C''$ is perpendicular to and bisects OA_4.

82. The distance of the centre of the inscribed circle from R (the intersection of perpendiculars from the angular points of the triangle ABC to the respective sides of $A''B''C''$) is equal to the diameter of the in-circle of triangle ABC.

For $A''B''$ is perpendicular to OC_4 and CR, and bisects them both. $CROC_4$ is an isosceles trapezoid, and $OR = CC_4 = 2r$.

83. If L be the middle point of OR, then LO equals r, and since WVU is perpendicular to OR at L, WVU becomes a tangent to the in-circle of ABC. The line WVU is also the radical axis, or the line of equal powers with respect to the circle about $A''B''C''$, and the centre O of the in-circle of ABC is the limiting point of the coaxal system. Each point on $A''B''$, and hence W, is equally distant from O and C_4. But W is also equally distant from R and C, therefore W is the centre of a circle passing through R, O, C_4, and C. Triangles $A_4B_4C_4$ and ABC are similar; O is the centre of the in-circle; and M' is the centre of the circumcircle of $A_4B_4C_4$. The angle OC_4M' equals $\frac{1}{2}(\alpha - \beta)$. Angle OCC_4, which is formed by the altitude from C and the bisector of the angle C, is also equal to $\frac{1}{2}(\alpha - \beta)$. OC_4 is a chord, and angle OCC_4 is an angle inscribed in the circle passing through ROC_4C; and since angle $M'C_4O$ equals angle OCC_4, $M'C_4$ is a tangent at the point C_4 to this circle. Hence $M'C_4$ is perpendicular to WC_4, or the

circles about $A_4B_4C_4$ and ROC_4C cut each other orthogonally. Similarly, circles from V and U, with radii equal to VB_4 and UA_4, will also pass through O and cut the circle about $A_4B_4C_4$ orthogonally.

The line joining the centres W, V, and U, is, therefore, the radical axis of circle about $A_4B_4C_4$ with respect to point O. The line joining the centres O and M' must be perpendicular to WVU. But OR was proved perpendicular to WVU (**77**), therefore M', O, and R are collinear.

84. The distance between M' and R is equal to the radius of the circumcircle of ABC.

Let the circle about $A_4B_4C_4$, whose centre is M', intersect the circle about ROC_4C, whose centre is W, at the point X. Since these circles cut each other orthogonally, $M'X$ will be perpendicular to WX, or $M'X$ is a tangent to circle whose centre is W, whence $\overline{M'X}^2 = M'O \cdot M'R$. We have, however, that $M'X = M'Q = OM$, hence

$$\overline{OM}^2 = M'O \cdot M'R \qquad (1).$$

OM is the distance between the centres of the inscribed and circumscribed circles of triangle ABC, and $\overline{OM}^2 = d^2 = r^2 - 2r\rho$, where r and ρ are the radii respectively of the circumscribed and inscribed circles.

Let the distance $M'R$ be denoted by x,

then $$M'O = x - 2\rho,$$

and $$M'R \cdot M'O = 2\rho(x - 2\rho).$$

Substituting in (1) above the values here given, we have

$$r^2 - 2r\rho = x(x - 2\rho),$$

or $$r(r - 2\rho) = x(x - 2\rho). \qquad (2)$$

From (2) it readily follows that $r = x$.

85. The perpendiculars from the vertices of the triangle $A'B'C'$ upon the respective sides of triangle $A''B''C''$ intersect at Z on the circumcircle of ABC (Fig. 20).

Let the perpendicular from A' to $B''C''$ and from B' to $A''C''$ intersect in Z, then the angle $A'ZB' = 180 - \measuredangle A''C''B''$

$= 90 + \frac{1}{2}\gamma$. And since $\measuredangle B'C'A' = 90 - \frac{1}{2}\gamma$, Z is on the same circumference as $A'B'C'$. We have now to prove that the perpendicular from C' to $A''B''$ passes also through Z, or that the line joining Z with C' is perpendicular to $A''B''$. Let B'_3 be the point of intersection ZB' and $A''C''$, and C'_3 the point of intersection of ZC' and $A''B''$; then

$$\measuredangle B'ZC' = \measuredangle B'A'C' = \measuredangle B''A''C'',$$

and $$\measuredangle B'_3A''C'_3 = 180 - \measuredangle C''A''B'' = 180 - \measuredangle B'ZC'.$$

Therefore Z, B'_3, A'', and C'_3 are concyclic.

And since $\measuredangle A''B'_3Z = 90°$, $\measuredangle A''C'_3Z$ also $= 90°$. The line ZC' is, therefore, perpendicular to $A''B''$.

Again, QA'' is perpendicular to $B'C'$, and $C'C'_3$ is perpendicular to $A''B''$, hence $\measuredangle C'_3A''Q = \measuredangle ZC'B'$, or $\measuredangle B''A''Q = \measuredangle ZA'B'$. The angle formed by ZA' and $A'B'$ is equal to the angle formed by QA'' and $A''B''$; or in the similar triangles $A'B'C'$ and $A''B''C''$, Z has the same relative position on the circumference of the circle about $A'B'C'$ with respect to the angular points of $A'B'C'$, as has Q on the circumference of the circle about $A''B''C''$ with respect to the angular points of $A''B''C''$.

86. We shall now prove that M, Q, and Z are collinear.

The triangles $A'B'C'$ and $A''B''C''$ are similar, the point O being the orthocentre in both triangles (**72**). The lines joining three points in $A'B'C'$ form the same angles as the lines joining three corresponding points in $A''B''C''$.

Therefore $$\measuredangle OM'Q = \measuredangle OMZ.$$

But $$\measuredangle OM'Q = \measuredangle OMQ,$$

hence $$\measuredangle OMZ = \measuredangle OMQ,$$

and the three points, M, Q, and Z are on the same straight line.

87. The perpendiculars from A to B_4C_4, from B to A_4C_4, and from C to A_4B_4 intersect at T on the circumference of the circle about ABC.

The proof for this is similar to that for the point Z in the previous discussion. Now T on the circumcircle of ABC has the same relative position with respect to the angular points of ABC, as Q has on the circumcircle of $A_4B_4C_4$ with respect to the angular points of $A_4B_4C_4$. HA_4 is perpendicular to BC, and CT is perpendicular to AB, therefore

$$\measuredangle B_4A_4H = \measuredangle BCT = \measuredangle BAT.$$

Also $$\measuredangle QB_4C_4 = \measuredangle ZBC = \measuredangle ZAC;$$

and since AC is parallel to QB_4, AZ is parallel to B_4C_4. In like manner we get BZ parallel to A_4C_4, and CZ parallel to A_4B_4.

88. The isogonal conjugates to AZ, BZ, CZ, with respect to triangle ABC are perpendicular to the line joining the orthocentre H with the fifth remarkable point Q of ABC. If through Z a parallel is drawn to AB to meet the circumcircle of ABC at Z', then the arc AZ equals arc BZ', being arcs between parallel chords. Therefore $\measuredangle ZCA = \measuredangle Z'CB$, and $Z'C$ is isogonal conjugate to ZC.

$$\measuredangle C_4HQ = \measuredangle CHQ = \measuredangle C_4B_4Q = \measuredangle CBZ = \measuredangle CZ'Z,$$

or $$\measuredangle CHQ = \measuredangle CZ'Z;$$

and since ZZ' is perpendicular to CH, therefore $Z'C$ must be perpendicular to HQ. The isogonal conjugates to ZA, ZB, and ZC are parallel to each other, and are all perpendicular to HQ. The Simson line to Z is perpendicular to the line isogonal conjugate to ZA, therefore it is parallel to HQ, and the Simson line to T must be perpendicular to HQ.

PLATE 1.

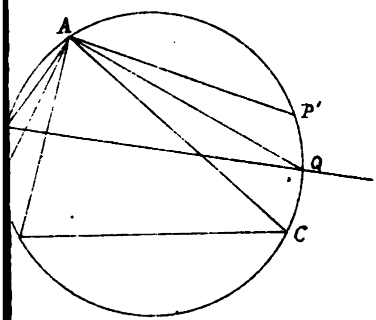

Fig. 3.

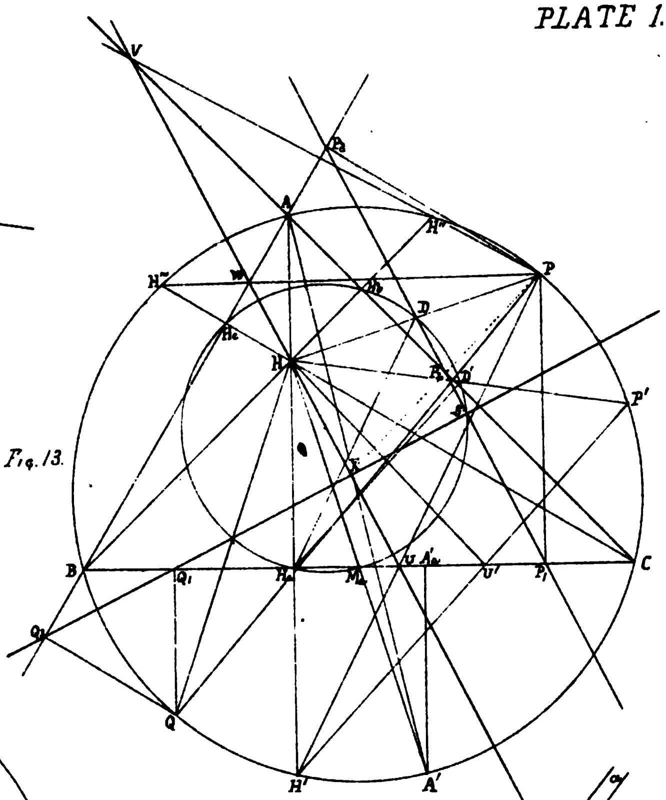

Fig. 13.

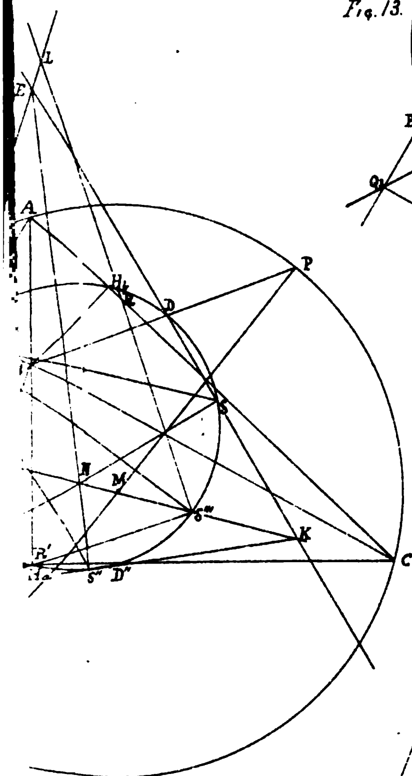

Fig. 16.

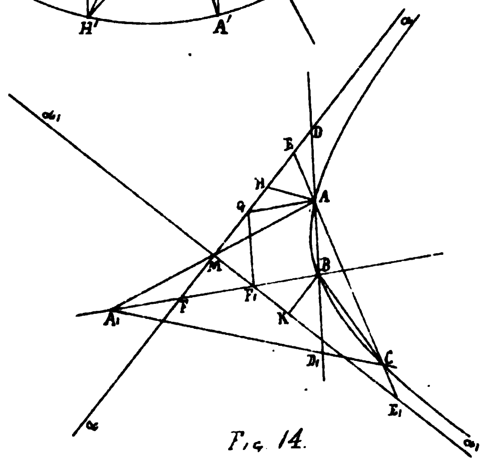

Fig. 14.

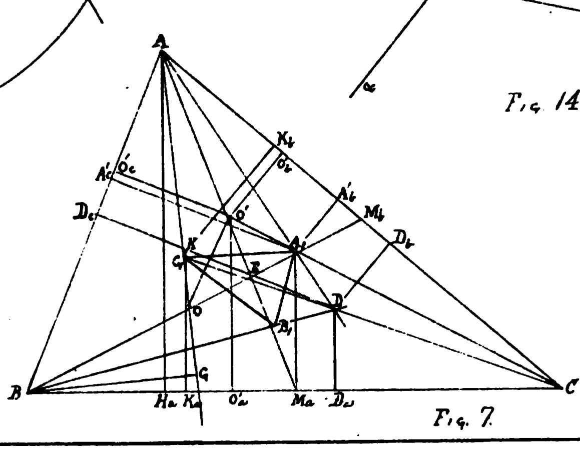

Fig. 7.

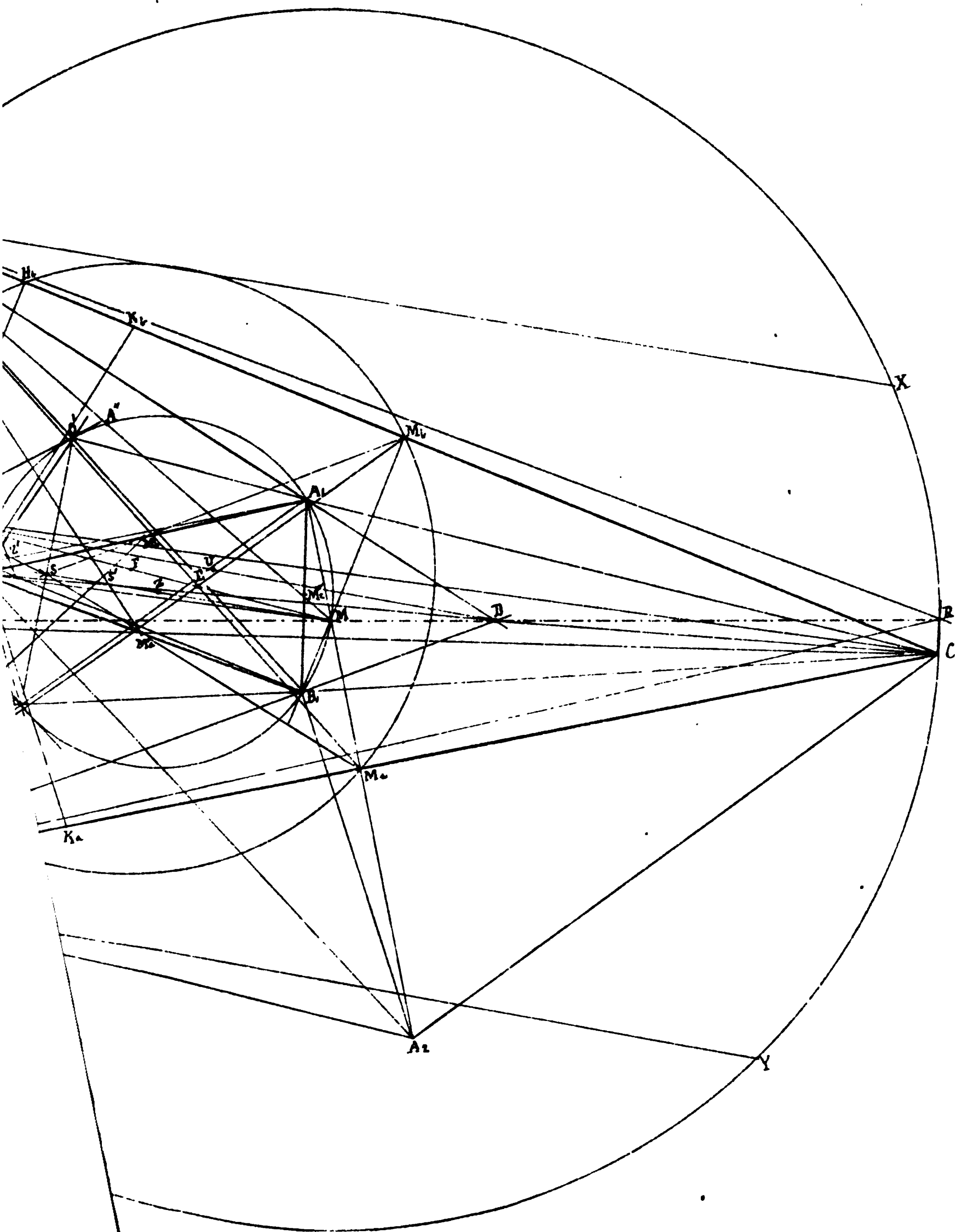

Fig. 15.

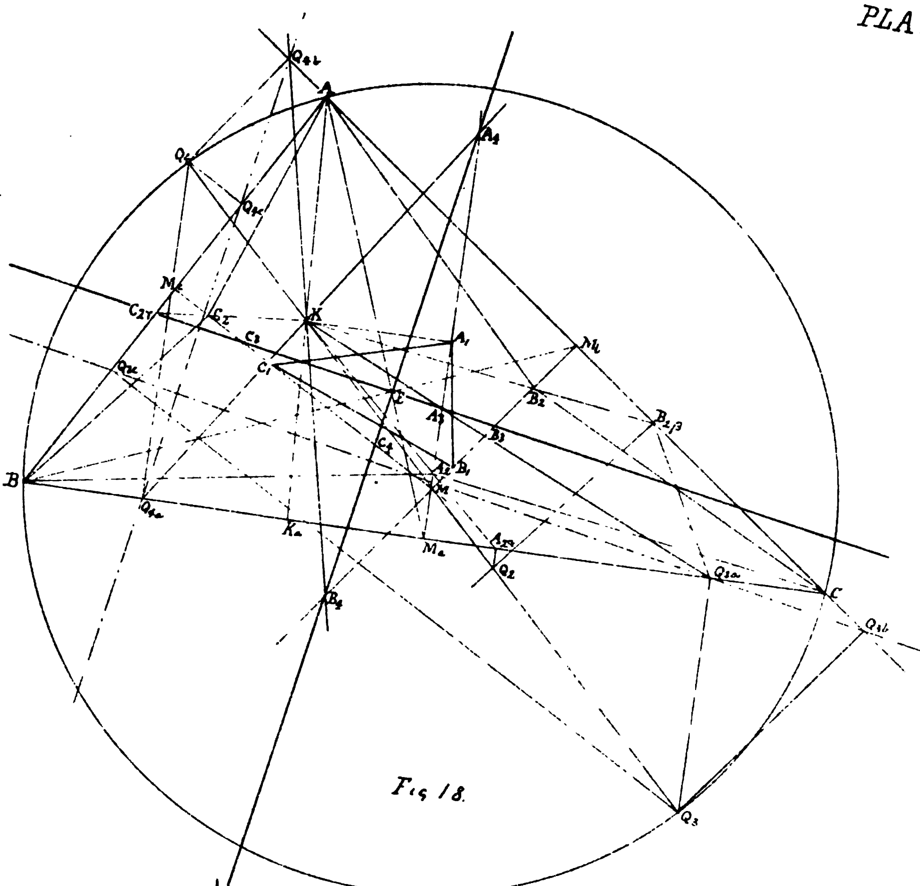

Fig 18.

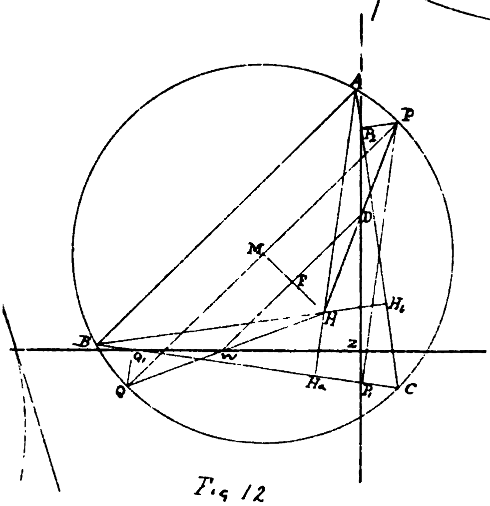

Fig 12

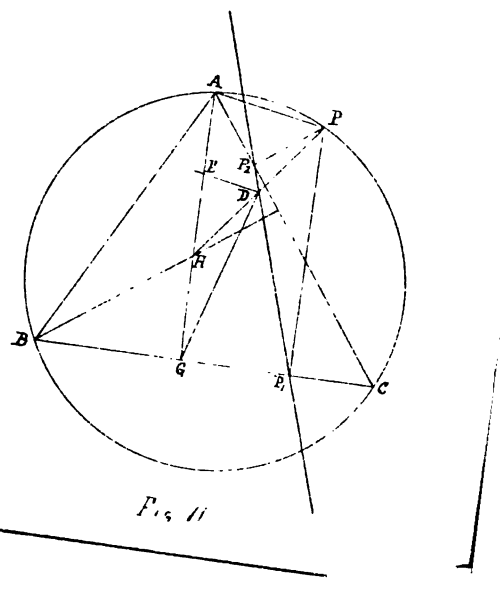

Fig 11

[illegible]